INFINITE FAMILIES OF HARMONIC SELF-MAPS OF SPHERES

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ABSTRACT. For each of the spheres $\mathbb{S}^n$, $n \geq 5$, we construct a new infinite family of harmonic self-maps, and prove that their members have Brouwer degree ± 1 or ± 3. These self-maps are obtained by solving a singular boundary value problem. As an application we show that for each of the special orthogonal groups $SO(4), SO(5), SO(6)$ and $SO(7)$ there exists two infinite families of harmonic self-maps.

1. INTRODUCTION

Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds and U a domain of M with piecewise C^1 boundary. The energy functional of φ over U is given by

$$E_U(\varphi) = \int_U |d\varphi|^2 \omega_g.$$

A smooth map $f : M \to N$ is called harmonic if it is a critical point of the energy functional. For the special case $M = N = \mathbb{S}^n$, where $\mathbb{S}^n$ is equipped with the standard metric, the Euler-Lagrange equations of the energy functional are given by the elliptic system

$$\Delta f + |df|^2 f = 0,$$

where Δ denotes the Laplace-Beltrami operator for the sphere $\mathbb{S}^n$. Finding solutions of this partial differential equation is difficult in general. By imposing symmetry conditions on the solution one can sometimes reduce this problem to finding solutions of an ordinary differential equation.

In this paper we restrict ourselves to self-maps of spheres which are equivariant with respect to the cohomogeneity one action

$$SO(m_0 + 1) \times SO(m_1 + 1) \times \mathbb{S}^{m_0 + m_1 + 1} \to \mathbb{S}^{m_0 + m_1 + 1}, \qquad (A, B, v) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} v.$$

In this case the Euler Lagrange equations reduce to the singular ordinary differential equation

$$\ddot{r}(t) = ((m_1 - m_0) \csc 2t - (m_0 + m_1) \cot 2t) \dot{r}(t) - m_1 \frac{\sin 2r(t)}{2 \cos^2 t} + m_0 \frac{\sin 2r(t)}{2 \sin^2 t}.$$

It was shown in [15] that each solution of this ordinary differential equation which satisfies $r(0) = 0$ and $r(\frac{\pi}{2}) = (2\ell + 1)\frac{\pi}{2}, \ell \in \mathbb{Z}$, yields a harmonic self-map of $\mathbb{S}^{m_0 + m_1 + 1}$.

The above ordinary differential equation and boundary value problem are henceforth referred to as (m_0, m_1)-ODE and (m_0, m_1)-BVP, respectively.

The goal of this paper is the construction of solutions of the (m_0, m_1)-BVP and the examination of their properties.

Initial value problem. In order to find solutions of the (m_0, m_1)-BVP we use a shooting method at the degenerate point $t = 0$. This is possible since for each $v \in \mathbb{R}$ there exists a unique solution r_v of the (m_0, m_1)-ODE with $r(0) = 0$ and $\dot{r}(0) = v$. This initial value problem is solved in Section 3.

The cases $2 \leq m_0 \leq 5$. We show that for $2 \leq m_0 \leq 5$ there exist infinitely many solutions of the (m_0, m_1)-BVP. These solutions are labeled by the number of intersections of r and $\frac{\pi}{2}$, the so-called *nodal number*.

Theorem A: Let $2 \leq m_0 \leq 5$ and $m_0 \leq m_1$. For each $k \in \mathbb{N}$ there exists a solution of the (m_0, m_1)-BVP with nodal number k.

For the special case that the multiplicities coincide, reflecting a solution of the (m, m)-BVP on the point $(\frac{\pi}{4}, \frac{\pi}{4})$ yields again a solution of the (m, m)-BVP. We use this fact to show that for $2 \leq m \leq 5$ there exist infinitely many solutions of the (m, m)-BVP with nodal number 0.

Theorem B: If $m_0 = m_1 =: m$ and $2 \leq m \leq 5$ there exists a countably infinite family of solutions of the (m, m)-BVP with nodal number 0.

Theorem A and B are proved in Section 4 and Section 6, respectively.

The cases $m_0 \geq 6$. We explain why for $m_0 \geq 6$ a construction analogous to that for the cases $2 \leq m_0 \leq 5$ is not possible. The reason is simply that for $m_0 \geq 6$ the nodal number is bounded from above.

Theorem C: Let r_v be the solution of the (m_0, m_1)-ODE with initial values $r(0) = 0$ and $\dot{r}(0) = v$. For $m_0 \geq 6$ the nodal number of r_v, $v \in \mathbb{R}$, is bounded from above by a constant which only depends on m_0 and m_1.

These results can be found in Section 5.

Limiting configuration. We prove that the solutions of the (m_0, m_1)-BVP converge against a limiting configuration when the initial velocity goes to infinity: we show that for large initial velocities r_v becomes arbitrarily close to $\frac{\pi}{2}$ on the interval $(0, \frac{\pi}{2})$.

Theorem D: For $t_0, t_1 \in (0, \frac{\pi}{2})$ and each $\epsilon > 0$ there exists a initial velocity v_0 such that $|r_v(t) - \frac{\pi}{2}| < \epsilon$ for all $t \in (t_0, t_1)$ and $v \geq v_0$.

This result can be found in Section 6.

Brouwer degree. Let r be a solution of the (m_0, m_1)-BVP. From Theorem 3.4 in [14] we deduce that the Brouwer degree of ψ_r is given by

$$\deg(\psi_r) = \begin{cases} 2\ell + 1 & \text{if } m_0 \text{ and } m_1 \text{ are even;} \\ -1 & \text{if } \ell, m_0 \text{ odd and } m_1 \text{ even;} \\ +1 & \text{otherwise,} \end{cases}$$

where ℓ is the integer determined by $r(\frac{\pi}{2}) = (2\ell + 1)\frac{\pi}{2}$. By a careful examination of the (m_0, m_1)-ODE we determine the possible $\ell \in \mathbb{Z}$ and thus obtain restrictions for the Brouwer degree.

Theorem E: For each solution r of the (m_0, m_1)-BVP, the Brouwer degree of ψ_r is ± 1 or ± 3.

Afterwards we prove that for large initial velocities the Brouwer degree of each solution of the (m_0, m_1)-BVP is ± 1.

Theorem F: There exists a $v_0 \in \mathbb{R}$ such that each solution of the (m_0, m_1)-BVP with initial velocity $v \geq v_0$ has Brouwer degree ± 1.

Numerical experiments indicate that there does not exist a solution r of the (m_0, m_1)-BVP such that the Brouwer degree of ψ_r is ± 3.

Theorems E and F are proved in Section 7.
Application. By combining a result of [15] with Theorems A and B we obtain the following theorem.

Theorem G: For each of the special orthogonal groups $SO(4), SO(5), SO(6)$ and $SO(7)$ there exists two infinite families of harmonic self-maps.

This theorem can be found in Section 6.

The paper is organized as follows: after giving some background information in Section 2, we provide the preliminaries in Section 3. In Section 4 we carry out the construction of infinitely many solutions of the (m_0, m_1)-BVP where $2 \leq m_0 \leq 5$ and thereby prove Theorem A. Afterwards, in Section 5, we deal with the cases $m_0 \geq 6$ and explain why an analogous construction to that of the cases $2 \leq m_0 \leq 5$ is not possible; we in particular prove Theorem C. In Section 6 we investigate the behavior of these solutions of the initial value problem with large initial velocities and prove Theorem D. As a byproduct we prove Theorem B. Form this theorem and Theorem A we deduce Theorem G. Finally, in Section 7 we give restrictions for the possible the Brouwer degrees of the solutions of the (m_0, m_1)-BVP; we in particular prove Theorems E and F.
Note that while the results of Section 3 are needed throughout the paper, Sections 3, 4, 5 and 6 can be read independently from each other.

2. Previous results

2.1. **Harmonic maps between spheres.** In this subsection we give a short and therefore incomplete survey on harmonic maps. The emphasize lies on harmonic maps between spheres.

The study of harmonic maps is an old problem which occupied generations of mathematicians. It received a significant boost in the last century by the paper of Eells and Sampson [8]. The basic question these authors examine is: does every homotopy class of maps between Riemannian manifolds admit a harmonic representative? For the special case that the target manifold is compact and all its sectional curvatures are nonnegative they gave a positive answer to this question. In contrast to this, for the case that the

target manifold also admits positiv sectional curvatures the answer to this question is only known in special cases. Even for maps between spheres this question is still open.

The paper of Eells and Sampson [8] was the starting point for a wealth of papers in which the classification and construction of harmonic maps between Riemannian manifolds has been pursued, see e.g. [3, 5, 6, 11, 12, 16] and the references therein. Due to the amount of existing results in the literature we will only mention those which have a direct relevance for this paper, we will in particular restrict ourselves to harmonic self-maps of spheres. For an introduction to harmonic maps we refer the reader to the book of Eells and Ratto [7].

As already mentioned in the introduction, additional symmetry assumptions can sometimes reduce the problem of constructing harmonic maps to finding solutions of an ordinary differential equation. For the general reduction theory we refer the reader to [7]. For the special case of harmonic maps between spheres there exists two basic reduction methods, the so-called harmonic Hopf and join constructions. Both of them we introduced by Smith [16]. While the Hopf construction is used for constructing homotopically nontrivial maps between spheres of large dimensions, the Join construction aims to the construction of homotopically nontrivial maps between spheres of small dimensions. Smith modified the Hopf construction and the Join construction such that it give a harmonic representative in the homotopy class of the Hopf map and join, respectively. Below we give a short survey of both reduction methods.

Recall that a map $f : \mathbb{S}^{p-1} \to \mathbb{S}^{q-1}$ with $p, q \geq 2$ is called an *eigenmap with eigenvalue* λ if $|df|^2 \equiv \lambda$. It is well-known that f is a harmonic eigenmap if and only if the components of f are harmonic polynomials of common degree d, which in particular implies $\lambda = d(p + d - 2)$. Furthermore, for non negative integers $p_1, p_2, q \geq 2$ a harmonic map $f : \mathbb{S}^{p_1-1} \times \mathbb{S}^{p_2-1} \to \mathbb{S}^{q-1}$ is called a *bi-eigenmap with eigenvalues* λ_1, λ_2 if for all $x_1 \in \mathbb{S}^{p_1-1}$ and all $x_2 \in \mathbb{S}^{p_2-1}$ the restrictions $f(\cdot, x_2)$ and $f(x_1, \cdot)$ are harmonic eigenmaps with eigenvalues λ_1 and λ_2, respectively.

Hopf Construction. In algebraic topology the Hopf construction of a map $f : \mathbb{S}^{p_1} \times \mathbb{S}^{p_2} \to \mathbb{S}^{q-1}$ is given by $H_f : \mathbb{S}^{p_1+p_2+1} \to \mathbb{S}^q$, $(x_1 \sin t, x_2 \cos t) \mapsto (f(x_1, x_2) \sin 2t, \cos 2t)$, where $x \in \mathbb{S}^{p_1+p_2+1}$ is written uniquely (with exemption of a set of measure zero) as $x = (x_1 \sin t, x_2 \cos t)$ for $x_1 \in \mathbb{S}^{p_1}$, $x_2 \in \mathbb{S}^{p_2}$ and $t \in [0, \frac{\pi}{2}]$. Smith [16] proved that

$$H(x_1 \sin t, x_2 \cos t) = (f(x_1, x_2) \sin u(t), \cos u(t)),$$

for some function $u : [0, \frac{\pi}{2}] \to [0, \pi]$ yields a harmonic map homotopic to H_f if f is a harmonic bi-eigenmap with eigenvalues $\lambda_1, \lambda_2 \in \mathbb{N}$ and u satisfies

$$\ddot{u}(t) + (p_1 \cot t - p_2 \tan t)\dot{u}(t) - \frac{1}{2}\left(\frac{\lambda_1}{\sin^2 t} + \frac{\lambda_2}{\cos^2 t}\right)\sin 2u(t) = 0,$$

with $u(0) = 0$ and $u(\frac{\pi}{2}) = \pi$. All constructions of harmonic maps based on this method crucially used that λ_2 is a positive integer. If we allow λ_2 to be negative, then for the special case $p_1 = \lambda_1 = m_0$, $p_2 = m_1$, $\lambda_2 = -m_1$ and $u = r$ the preceding ordinary differential equation coincides with the (m_0, m_1)-ODE. The boundary condition at $\frac{\pi}{2}$ is however not of the form of that of the (m_0, m_1)-BVP.

Join Construction. The join of two homogeneous polynomials $f_i : \mathbb{S}^{p_i} \to \mathbb{S}^{q_i}$, $i \in \{1, 2\}$, is given by $J_{f_1, f_2} : \mathbb{S}^{p_1+p_2+1} \to \mathbb{S}^{q_1+q_2+1}$, $(x_1 \sin t, x_2 \cos t) \mapsto (f_1(x_1) \sin t, f_2(x_2) \cos t)$, where x_1 and x_2 are defined as above. Smith [16] proved that whenever f_1, f_2 are harmonic eigenmaps with eigenvalues λ_1, λ_2, then the ansatz

$$J(x_1 \sin t, x_2 \cos t) = (f_1(x_1) \sin u(t), f_2(x_2) \cos u(t)),$$

for some function $u : [0, \frac{\pi}{2}] \to [0, \frac{\pi}{2}]$, yields a harmonic map homotopic to J_{f_1, f_2} if u satisfies

$$\ddot{u}(t) + (p_1 \cot t - p_2 \tan t)\dot{u}(t) - \tfrac{1}{2}\left(\tfrac{\lambda_1}{\sin^2 t} - \tfrac{\lambda_2}{\cos^2 t}\right)\sin 2u(t) = 0,$$

$u(0) = 0$, $u(\frac{\pi}{2}) = \frac{\pi}{2}$ and $0 \le u \le \frac{\pi}{2}$. All constructions of harmonic maps based on this method crucially used that u only attains values between 0 and $\frac{\pi}{2}$. For the special case $p_1 = \lambda_1 = m_0$, $p_2 = \lambda_2 = m_1$ and $u = r$ the preceding ordinary differential equation coincides with the (m_0, m_1)-ODE. Most solutions we found numerically however do not satisfy $0 \le u \le \frac{\pi}{2}$, meaning that the constructions in the literature are not suited to our boundary value problem.

2.2. **Harmonic maps between cohomogeneity one manifolds.** In this subsection we explain in which context the (m_0, m_1)-BVP arises.

The equivariant homotopy classes of equivariant self-maps of compact cohomogeneity one manifolds whose orbit space is a closed interval form an infinite family. In [15] the problem of finding harmonic representatives of these homotopy classes was reduced to solving singular boundary value problems for nonlinear second order ordinary differential equations.

Below we consider the special case of isometric cohomogeneity one actions $G \times \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$ where G is a subgroup of $\mathrm{SO}(n+2)$. The orbits of any such action yield an isoparametric foliation of the sphere. The data of such a foliation are the numbers g of distinct principal curvatures of the isoparametric hypersurfaces and the multiplicities $m_0, \ldots, m_{g-1}$. If g is odd, all multiplicities coincide $m := m_0 = \ldots = m_{g-1}$. If g is even, we have $m_0 = \ldots = m_{g-2}$ and $m_1 = m_3 = \ldots = m_{g-1}$. Münzner [13] proved $g \in \{1, 2, 3, 4, 6\}$. For $g = 1$ and $g = 2$ there are no restrictions for the multiplicities; for $g = 3$ all multiplicities coincide and are either given by $1, 2, 4$ or 8 [4]; for $g = 4$ the possible multiplicities can be found in [9]; for $g = 6$ all multiplicities coincide and are given by 1 or 2 [1].

Let H denote the principal isotropy group (along one normal geodesic) of the cohomogeneity one action $G \times \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$. It was shown in [15] that the map

$$\psi_r : G/H \times \,]0, \pi/g[\, \to G/H \times \mathbb{R}, \qquad (gH, t) \to (gH, r(t)),$$

is harmonic if and only if r solves the boundary value problem

$$(2.1) \qquad \ddot{r}(t) = -\tfrac{1}{4\sin^2 gt}\Big\{\big(g(m_0 + m_1)\sin 2gt + 2g(m_0 - m_1)\sin gt\big)\dot{r}(t)$$
$$-g(g-2)\sin 2(r-t)\big(m_0 + m_1 + (m_0 - m_1)\cos gt\big)$$
$$-2g\sin(2(r-t)+gt)\big((m_0 + m_1)\cos gt + m_0 - m_1\big)\Big\},$$

with $\lim_{t \to 0} r(t) = 0$ and $\lim_{t \to \pi/g} r(t) = (gk+1)\frac{\pi}{g}$ for a $k \in \mathbb{Z}$.

For $g = 2$ this boundary value problem reduces to the (m_0, m_1)-BVP.

2.3. What is known?

In this subsection we explain which of the boundary values (2.1) are discussed in former papers.

The case $g = 1$ was considered by Bizon and Chmaj: in [3] they studied the boundary value problem

$$\ddot{r}(t) = \tfrac{1}{2} m \csc^2 t \big(\sin 2r(t) - \sin 2t \cdot \dot{r}(t) \big), \qquad r(0) = 0, \quad r(\pi) = \ell\pi, \, \ell \in \mathbb{Z}.$$

This is the boundary value problem associated to the cohomogeneity one actions whose orbits are homogeneous isoparametric hypersurfaces in spheres with one principal curvature of multiplicity m. Since Bizon and Chmaj were seeking for point or reflection symmetric solutions, they could use a shooting method at the regular point $t = \frac{\pi}{2}$ to construct solutions with one of these additional symmetries. Thereby they proved that for each of the cases $2 \le m \le 5$ there exists an infinite family of harmonic self-maps of $\mathbb{S}^{2m+1}$.

Although Theorem A has a certain similarity to the result of Bizon and Chmaj, the methods to prove it are different. Clearly, the fact that in the case of the (m_0, m_1)-BVP we have to deal with two possibly distinct multiplicities makes the situation more complicated. But even if the multiplicities coincide there a more complications: numerical experiments indicate that up to finitely many exceptions the solutions of the (m_0, m_1)-BVP are neither point nor reflection symmetric. This means we have to consider a shooting method at a singular point rather than at a regular one.

Baird [2] derived the boundary value problem for $g = 4$; see equation (5.3.25) in [2]. Since the methods used to construct some individual solutions of these equation differ from those used in this paper we refer the reader to the book of Baird for more details.

3. Preliminaries

This section serves as preparation for the following sections. In the first subsection we introduce another variable which we will use throughout this paper. Afterwards, in the second subsection, we prove that for each $v \in \mathbb{R}$ there exists a unique solution r_v of the (m_0, m_1)-ODE with $r_v(t)_{|t=0} = 0$. Finally, in the third subsection we provide several restrictions for solutions r of the (m_0, m_1)-ODE.

3.1. The variable x.

Throughout this paper we will use not only the variable t but also the variable $x = \log(\tan t)$. In terms of the variable $x = \log(\tan t)$ the (m_0, m_1)-BVP is given by

$$r''(x) = \tfrac{1}{2} \big((m_0 + m_1 - 2) \tanh x + m_1 - m_0 \big) r'(x)$$
$$- \tfrac{1}{4} \big((m_0 + m_1) \tanh x + m_1 - m_0 \big) \sin 2r(x),$$

with $\lim_{x \to -\infty} r(x) = 0$ and $\lim_{x \to \infty} r(x) = (2\ell + 1)\frac{\pi}{2}$, $\ell \in \mathbb{Z}$. It is convenient to introduce the functions $\alpha_{m_0, m_1}, \beta_{m_0, m_1} : \mathbb{R} \to \mathbb{R}$ by

$$\alpha_{m_0, m_1} : x \mapsto \tfrac{1}{2} \big((m_0 + m_1 - 2) \tanh x + m_1 - m_0 \big)$$

and $\beta_{m_0,m_1} = \frac{1}{2}\alpha_{m_0+1,m_1+1}$ such that the (m_0, m_1)-ODE is given by

$$r''(x) - \alpha_{m_0,m_1}(x)r'(x) + \beta_{m_0,m_1}(x)\sin 2r(x) = 0.$$

We have $\alpha_{1,1} \equiv 0$. If $m_1 > 1$ then $\alpha_{1,m_1} > 0$ with $\lim_{x \to -\infty} \alpha_{1,m_1}(x) = 0$.

Notation 3.1: For $m_0 > 1$, we denote by $Z^\alpha_{m_0,m_1} \in \mathbb{R}$ the unique zero of the function α_{m_0,m_1}. If $m_0 = 1$ and $m_1 > 1$, then we set $Z^\alpha_{1,m_1} = -\infty$. Furthermore, we denote by $Z^\beta_{m_0,m_1} \in \mathbb{R}$ the unique zero of the function β_{m_0,m_1}.

3.2. **Initial value problem.** In order to solve the initial value problem at $t = 0$ we use a theorem of Malgrange in the version that can be found in [10].

Theorem of Malgrange (Theorem 4.7 in [10]): *Consider the singular initial value problem*

(3.1) $$\dot{y} = \tfrac{1}{t}M_{-1}(y) + M(t, y), \qquad y(0) = y_0,$$

where y takes values in $\mathbb{R}^k$, $M_{-1} : \mathbb{R}^k \to \mathbb{R}^k$ is a smooth function of y in a neighborhood of y_0 and $M : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is smooth in t, y in a neighborhood of $(0, y_0)$. Assume that

(i) $M_{-1}(y_0) = 0$,
(ii) $h\,\mathrm{Id} - d_{y_0}M_{-1}$ is invertible for all $h \in \mathbb{N}$, $h \geq 1$.

Then there exists a unique solution $y(t)$ of (3.1). Furthermore y depends continuously on y_0 satisfying (i) and (ii).

Lemma 3.2: *For each $v \in \mathbb{R}$ the initial value problem $r(t)_{|t=0} = 0, \dot{r}(0) := \frac{d}{dt}r(t)_{|t=0} = v$ has a unique solution r_v. The functions r_v and $\frac{d}{dt}r_v$ depend continuously on v. Furthermore, there exists no $t_0 \in \mathbb{R}$ with $r_v(t_0) = \frac{\pi}{2}$ and $\dot{r}_v(t_0) = 0$.*

Proof. We introduce the variable $s = t^2$ and the operator $\theta = s\frac{d}{ds}$. Clearly, $\frac{d}{dt} = \frac{2}{\sqrt{s}}\theta$ and $\frac{d^2}{dt^2} = -\frac{2}{s}\theta + \frac{4}{s}\theta^2$. In terms of s and θ the ODE is given by

$$\theta^2 r = \tfrac{1}{2}\theta r - \frac{\sqrt{s}}{2\sin(2\sqrt{s})}\left((m_0 + m_1)\cos(2\sqrt{s}) + (m_0 - m_1)\right)\theta r$$
$$+ \tfrac{s}{2^2}\csc^2(2\sqrt{s})(m_0 - m_1 + (m_0 - m_1)\cos(2\sqrt{s}))\sin 2r.$$

Next we rewrite this ODE as a first order system

$$\theta(r) = \theta r, \qquad \theta(\theta r) = \psi$$

and we compute the partial derivatives of the right hand sides with respect to r and θr at $s = 0$. We thus obtain

$$\begin{pmatrix} \frac{\partial}{\partial r}\theta r & \frac{\partial}{\partial \theta r}\theta r \\ \frac{\partial}{\partial r}\psi & \frac{\partial}{\partial \theta r}\psi \end{pmatrix}_{|s=0} = \begin{pmatrix} 0 & 1 \\ \frac{1}{4}m_0 & \frac{1}{2}(1 - m_0) \end{pmatrix}.$$

Since the eigenvalues of this matrix are given by $\frac{1}{2}$ and $-\frac{m_0}{2}$, the Theorem of Malgrange states that a formal power series solution of this equation converges to a unique solution in a neighborhood of $s = 0$. This solution depends continuously on v. $\square$

Introduce the new variable $u = \frac{\pi}{2} - t$. Similarly as in the above lemma one proves for each $v \in \mathbb{R}$ the initial value problem $r(u)_{|u=0} = (2k + 1)\frac{\pi}{2}, \frac{d}{du}r(u)_{|u=0} = v$ has a unique solution.

3.3. **Restrictions for** r. In this subsection we prove that there exists a constant $d^-_{m_0,m_1} \in \mathbb{R}$ such that for each solution r of the (m_0, m_1)-ODE with $\lim_{x \to -\infty} r(x) = 0$ either $0 \leq r(x) \leq \pi$ or $-\pi \leq r(x) \leq 0$, for all $x \leq d^-_{m_0,m_1}$. Furthermore, we show that if r is a solution of the (m_0, m_1)-BVP then there exist $d^+_{m_0,m_1} \in \mathbb{R}$ and $\ell_0 \in \mathbb{Z}$ such that $(2\ell_0 + 1)\frac{\pi}{2} \leq r(x) \leq (2\ell_0 + 3)\frac{\pi}{2}$ for all $x \geq d^+_{m_0,m_1}$. In the following picture one can find a sketch of one solution with $\ell_0 = 0$. Since we do not know anything about the behavior of the solutions in the interval $[d^-_{m_0,m_1}, d^+_{m_0,m_1}]$ the line is dotted in this region.
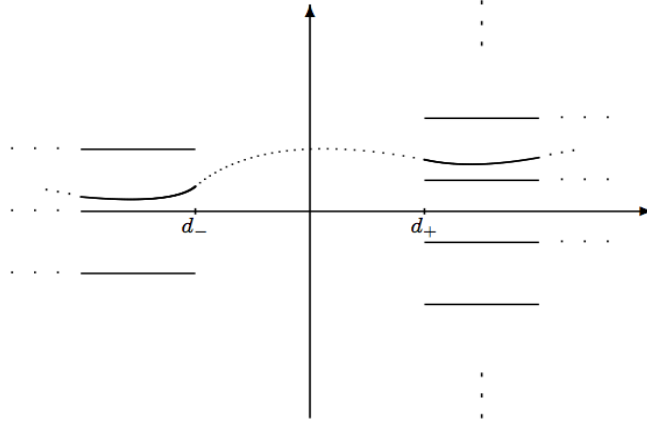
3.3.1. *Behavior for large positive* x. An important tool throughout this subsection is the map $W^r_{m_0,m_1} : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \frac{1}{2}r'(x)^2 + \beta_{m_0,m_1}(x)\sin^2 r(x)$, which turns out to be a Lyapunov function.

Lemma 3.3: *Either* $W^r_{m_0,m_1}$ *increases strictly on* $[Z^\alpha_{m_0,m_1}, \infty)$ *or* $W^r_{m_0,m_1} \equiv 0$. *If the latter case occurs then* r *is constant.*

Proof. By using the (m_0, m_1)-ODE we get

$$\frac{d}{dx}W^r_{m_0,m_1}(x) = \alpha_{m_0,m_1}(x)r'(x)^2 + \frac{m_0+m_1}{4\cosh^2 x}\sin^2 r(x).$$

Thus $\frac{d}{dx}W^r_{m_0,m_1}(x) \geq 0$ for $x \geq Z^\alpha_{m_0,m_1}$. If $W^r_{m_0,m_1}$ increases strictly there is nothing to prove. Hence we may assume that there exists a point $x_0 \geq Z^\alpha_{m_0,m_1}$ such that $\frac{d}{dx}W^r_{m_0,m_1}(x_0) = 0$, which implies $r(x_0) = \ell_0\pi$ for an $\ell_0 \in \mathbb{Z}$.

If $(m_0, m_1) \neq (1,1)$ and $x_0 > Z^\alpha_{m_0,m_1}$, then $\frac{d}{dx}W^r_{m_0,m_1}(x_0) = 0$ also yields $r'(x_0) = 0$. Hence, by the theorem of Picard-Lindelöf we have $r \equiv \ell_0\pi$ and thus $W^r_{m_0,m_1} \equiv 0$.

Hence only the cases $(m_0, m_1) = (1,1)$ or $x_0 = Z^\alpha_{m_0,m_1}$ is satisfied remain to consider. If also $r'(x_0) = 0$ then the same argument as above yields $W^r_{m_0,m_1} \equiv 0$. Finally, if $r'(x_0) \neq 0$ there exists a connected neighborhood $U \subset [Z^\alpha_{m_0,m_1}, \infty)$ of x_0 such that $r'(x) \neq 0$ and $r(x) \neq k\pi$ for all $x \in U - \{x_0\}$. Consequently, $\frac{d}{dx}W^r_{m_0,m_1}(x) > 0$ for all $x \in U - \{x_0\}$ and thus $W^r_{m_0,m_1}$ increases strictly. $\qquad\square$

Using the above lemma we show that on the interval $[Z^\alpha_{m_0,m_1}, \infty)$ the first derivative of any solution r of the (m_0, m_1)-BVP is bounded.

Lemma 3.4: *For any solution r of the (m_0, m_1)-ODE with $\lim_{x\to\infty} r(x) = (2\ell + 1)\frac{\pi}{2}$, $\ell \in \mathbb{Z}$, we have $|r'(x)| \leq \sqrt{m_1}$ for $x \geq Z^{\beta}_{m_0,m_1}$ and $|r'(x)| \leq \sqrt{m_1 + 1}$ for $x \geq Z^{\alpha}_{m_0,m_1}$.*

Proof. If r is constant there is nothing to prove. Hence we may assume that r is non-constant. Using the assumption $\lim_{x\to\infty} r(x) = (2\ell + 1)\frac{\pi}{2}$, $\ell \in \mathbb{Z}$, we obtain $\lim_{x\to\infty} W^r_{m_0,m_1}(x) = \frac{m_1}{2}$. Since for $x \geq Z^{\beta}_{m_0,m_1}$ both summands in the definition of $W^r_{m_0,m_1}(x)$ are positive and $W^r_{m_0,m_1}$ increases strictly on the interval $[Z^{\alpha}_{m_0,m_1}, \infty)$, we get $\frac{m_1}{2} \geq W^r_{m_0,m_1}(x) \geq \frac{1}{2} r'(x)^2$ for $x \geq Z^{\beta}_{m_0,m_1}$, whence the first claim. Furthermore,

$$\frac{m_1}{2} \geq W^r_{m_0,m_1}(x) \geq \frac{1}{2} r'(x)^2 + \beta_{m_0,m_1}(Z^{\alpha}_{m_0,m_1}) \sin^2 r(x) \geq \frac{1}{2} r'(x)^2 - \frac{1}{2}$$

for all $x \in [Z^{\alpha}_{m_0,m_1}, Z^{\beta}_{m_0,m_1}]$, whence the second claim. $\qquad\square$

In the following lemma we prove that each solution of the (m_0, m_1)-ODE either converges to $\pm$ infinity or to $(2\ell_0 + 1)\frac{\pi}{2}$, $\ell_0 \in \mathbb{Z}$, as x converges to infinity.

Lemma 3.5: *Let $m_1 \geq 2$ and r be a non-constant solution of the (m_0, m_1)-ODE. Either there exists $\ell_0 \in \mathbb{Z}$ such that $\lim_{x\to\infty} r(x) = (2\ell_0 + 1)\frac{\pi}{2}$ or $\lim_{x\to\infty} r(x) = \pm\infty$.*

Proof. If $\lim_{x\to\infty} r'(x) = 0$ then by Lemma 3.3 $\lim_{x\to\infty} W^r_{m_0,m_1}(x) = \frac{m_1}{2} \lim_{x\to\infty} \sin^2 r(x)$ exists. Thus $L := \lim_{x\to\infty} r(x)$ exists and is finite and the (m_0, m_1)-ODE implies $\lim_{x\to\infty} r''(x) = -\frac{m_1}{2} \sin 2L$. Consequently, $L = \ell_0\pi$ or $L = \ell_0\pi + \frac{\pi}{2}$ for an $\ell_0 \in \mathbb{Z}$.

If $L = \ell_0\pi$ we get $\lim_{x\to\infty} W^r_{m_0,m_1}(x) = 0$. Since $W^r_{m_0,m_1}(0) \geq 0$, Lemma 3.3 implies $W^r_{m_0,m_1} \equiv 0$. However, this in turn yields $r \equiv \ell_0\pi$ which contradicts the assumption that r is non-constant. Hence $\lim_{x\to\infty} r'(x) = 0$ implies $\lim_{x\to\infty} r(x) = \ell_0\pi + \frac{\pi}{2}$.

When $\lim_{x\to\infty} r'(x) \neq 0$ we get $\lim_{x\to\infty} \frac{d}{dx} W^r_{m_0,m_1}(x) \neq 0$. Since $\frac{d}{dx} W^r_{m_0,m_1}(x) \geq 0$ for all $x \geq Z^{\alpha}_{m_0,m_1}$ we get $\lim_{x\to\infty} W^r_{m_0,m_1}(x) = \infty$ and thus $\lim_{x\to\infty} r'(x)^2 = \infty$. Hence, for every $\epsilon \in \mathbb{R}_+$ there exists a point $x_0 \in \mathbb{R}$ such that $|r'(x)| > \epsilon$ for all $x > x_0$. Thus $\lim_{x\to\infty} r(x) = \pm\infty$. $\qquad\square$

So far we have not find any restrictions for the possible $\ell_0 \in \mathbb{Z}$ - this will be done in Section 7. In the following lemma we improve the result of Lemma 3.4.

Lemma 3.6: *For $m_1 \geq 2$ let $B = \frac{m_1}{2(m_1 - 1)}$. There exists $c_{m_0,m_1} \in \mathbb{R}$ such that*

    (i) if $r'(x_0) > B$ for an $x_0 > c_{m_0,m_1}$ then $\lim_{x\to\infty} r(x) = \infty$,

    (ii) if $r'(x_0) < -B$ for an $x_0 > c_{m_0,m_1}$ then $\lim_{x\to\infty} r(x) = -\infty$.

Proof. We introduce the quotient function $q_{m_0,m_1} : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}, x \mapsto \frac{\beta_{m_0,m_1}(x)}{\alpha_{m_0,m_1}(x)}$. If $m_0 < m_1$ then q_{m_0,m_1} increases strictly on the interval $I = (Z^{\alpha}_{m_0,m_1}, \infty)$ and satisfies $\text{Im}(q_{m_0,m_1|I}) = (-\infty, B)$. The unique solution $x > Z^{\alpha}_{m_0,m_1}$ of $q_{m_0,m_1}(x) = -B$ is denoted by c_{m_0,m_1}. If $m_0 = m_1 =: m$ then $q_{m,m} = B$ and we set $c_{m,m} = 0$.

The strategy for the proof of (i) is the following: we show that the existence of a point $x_0 > c_{m_0,m_1}$ with $r'(x_0) > B$ implies $r''(x) > 0$ for all $x \geq x_0$. Consequently, $r'(x) \geq r'(x_0) > B > 0$ for all $x \geq x_0$ and thus $\lim_{x\to\infty} r(x) = \infty$.

First we prove $r''(x_0) > 0$: since $|q_{m_0,m_1}(x)| \leq B$ for $x \geq c_{m_0,m_1}$ we have

$$q_{m_0,m_1}(x) \sin 2r(x) \leq B \quad \text{for} \quad x \geq c_{m_0,m_1}.$$

Thus $B < r'(x_0)$ yields $q_{m_0,m_1}(x_0) \sin 2r(x_0) < r'(x_0)$. The (m_0, m_1)-ODE implies

$$q_{m_0,m_1}(x) \sin 2r(x) < r'(x) \quad \Leftrightarrow \quad r''(x) > 0 \quad \text{for all } x > Z^{\alpha}_{m_0,m_1}.$$

Since $x_0 > c_{m_0,m_1} \geq Z^{\alpha}_{m_0,m_1}$, we thus obtain $r''(x_0) > 0$.

Next suppose that there exists a point $x_1 > x_0$ such that $r''(x_1) = 0$ and $r''(x) > 0$ for all $x \in [x_0, x_1)$. Hence, $r'(x) > r'(x_0) > B$ for all $x \in (x_0, x_1)$. Since r' is a continuous function we thus obtain $r'(x_1) > B \geq q_{m_0,m_1}(x_1) \sin 2r(x_1)$. This inequality is equivalent to $r''(x_1) > 0$ contradicting our assumption. Therefore $r''(x) > 0$ for all $x \geq x_0$ and thus $r'(x) \geq r'(x_0) \geq B$ for all $x \geq x_0$. Hence, $\lim_{x \to \infty} r(x) = \infty$.

The second statement is obtained by the first by considering $-r$. $\qquad\square$

The next lemma states that for large enough x the graph of each solution of the (m_0, m_1)-BVP is contained in a stripe of height π.

Lemma 3.7: *For $m_1 \geq 2$ there exists $d^{+}_{m_0,m_1} \in \mathbb{R}$ such that one of the following three cases arises:*

(i) *there exists an $\ell_0 \in \mathbb{Z}$ with $(2\ell_0 + 1)\frac{\pi}{2} \leq r(x) \leq (2\ell_0 + 3)\frac{\pi}{2}$ for all $x \geq d^{+}_{m_0,m_1}$. Then either $r \equiv (2\ell_0 + 1)\frac{\pi}{2}, (2\ell_0 + 2)\frac{\pi}{2}, (2\ell_0 + 3)\frac{\pi}{2}$ or, if r is non-constant, $\lim_{x \to \infty} r(x) = (2\ell_0 \pm 1)\frac{\pi}{2}$.*

(ii) *there exist $x_0 \geq d^{+}_{m_0,m_1}$ and $\ell_0 \in \mathbb{Z}$ such that $r(x_0) = (2\ell_0 + 3)$ and $r'(x_0) > 0$. Then $\lim_{x \to \infty} r(x) = \infty$.*

(iii) *there exist $x_0 \geq d^{+}_{m_0,m_1}$ and $\ell_0 \in \mathbb{Z}$ such that $r(x_0) = (2\ell_0 + 3)$ and $r'(x_0) < 0$. Then $\lim_{x \to \infty} r(x) = -\infty$.*

Proof. The equation $2\beta_{m_0,m_1}(x) = B^2$ has a unique solution which we denote by $d^{+}_{m_0,m_1}$.

Assume that there exist $\ell_0 \in \mathbb{Z}$ and $x_0 \geq d^{+}_{m_0,m_1} \in \mathbb{R}$ such that $r(x_0) = (2\ell_0 + 3)\frac{\pi}{2}$. If r is a solution of the (m_0, m_1)-ODE, so is $r + j\pi$ for each $j \in \mathbb{Z}$. Thus we may without loss of generality assume $\ell_0 = -1$. If $r'(x_0) = 0$ the theorem of Picard-Lindelöf implies $r \equiv \frac{\pi}{2}$. Hence only the case $r'(x_0) \neq 0$ remains to consider. We can assume without loss of generality $r'(x_0) > 0$: if $r'(x_0) < 0$ we consider $-r + \pi$ instead of r.

Since $x_0 \geq d^{+}_{m_0,m_1} > Z^{\alpha}_{m_0,m_1}$, we get $\alpha_{m_0,m_1}(x_0) > 0$. Using $r'(x_0) > 0$ the (m_0, m_1)-ODE thus yields $r''(x_0) > 0$. In what follows we assume that there exists no point $x_1 > x_0$ with $r(x_1) = \pi$ and $r'(x_1) \geq 0$.

If $r''(x) \geq 0$ for all $x \geq x_0$, then $r'(x) \geq r'(x_0) > 0$ for all $x \geq x_0$. However, this implies the existence of a point $x_1 > x_0$ with $r(x_1) = \pi$ and $r'(x_1) \geq 0$, which contradicts our assumption. Consequently, there exists a point $y > x_0$ with $r''(y) = 0$ such that $r''(x) > 0$ and $\frac{\pi}{2} \leq r(x) < \pi$ for all $x \in [x_0, y)$. Thus $r'(x) > r'(x_0) > 0$ for all $x \in (x_0, y)$. Hence continuity of r and r' yield $\frac{\pi}{2} \leq r(y) \leq \pi$ and $r'(y) \geq r'(x_0) > 0$, respectively. Since $y > x_0 \geq d^{+}_{m_0,m_1} > Z^{\beta}_{m_0,m_1} \geq Z^{\alpha}_{m_0,m_1}$ we get $\alpha_{m_0,m_1}(y) > 0$ and $\beta_{m_0,m_1}(y) > 0$. Using the (m_0, m_1)-ODE we thus obtain $r''(y) > 0$, which contradicts our assumption.

Therefore there exists $x_1 > x_0$ with $r(x_1) = \pi$ and $r'(x_1) \geq 0$. Thus Lemma 3.3 yields

$$W^{r}_{m_0,m_1}(x_1) - W^{r}_{m_0,m_1}(x_0) = \tfrac{1}{2}(r'(x_1)^2 - r'(x_0)^2) - \beta_{m_0,m_1}(x_0) \geq 0.$$

Using $x_0 \geq d_{m_0,m_1}^+ \geq c_{m_0,m_1}$ and the fact that β_{m_0,m_1} is an increasing function, we get $r'(x_1)^2 > 2\beta_{m_0,m_1}(x_0) \geq B^2$. Since $r'(x_1) \geq 0$ we obtain $r'(x_1) > B$. Hence Lemma 3.6 implies $\lim_{x\to\infty} r(x) = \infty$. Consequently, either $\lim_{x\to\infty} r(x) = \pm\infty$ or $(2\ell_0 + 1)\frac{\pi}{2} \leq r(x) \leq (2\ell_0 + 3)\frac{\pi}{2}$ for $x \geq d_{m_0,m_1}^+$. Now the claim follows from Lemma 3.5. $\qquad\square$

As already mentioned in Section 2 there are several properties which the solutions of the (m_0, m_1)-BVP and the solutions of the boundary value problem considered by Bizon and Chmaj in [3] have in common. Nevertheless, there are some decisive differences. One of them is the following: while Bizon and Chmaj construct infinitely many solutions which are symmetric with respect to the y-axis, this is not possible for the (m_0, m_1)-BVP. Indeed, the previous lemma implies that any solution of the (m_0, m_1)-BVP which is symmetric with respect to the y-axis, is constant.

Theorem 3.8: *Let $m_1 \geq 2$ and r be non-constant with $\lim_{x\to-\infty} r(x) = 0$. Then r is either a solution of the (m_0, m_1)-BVP or satisfies $\lim_{x\to\infty} r(x) = \pm\infty$.*
If r is a solution of the (m_0, m_1)-BVP then there exist $d_{m_0,m_1}^+ \in \mathbb{R}$ and $\ell_0 \in \mathbb{Z}$ such that $(2\ell_0+1)\frac{\pi}{2} \leq r(x) \leq (2\ell_0+3)\frac{\pi}{2}$ for all $x \geq d_{m_0,m_1}^+$. Furthermore, in this case there exists a point $x_0 > d_{m_0,m_1}^+$ such that either $r'(x) \geq 0$ or $r'(x) \leq 0$, for all $x > x_0$.

Proof. To prove the claim, it is sufficient to show that in case (i) of Lemma 3.7 there exists a point $x_0 > d_{m_0,m_1}^+$ such that either $r'(x) \geq 0$ or $r'(x) \leq 0$, for all $x > x_0$.

Since r is non-constant we may assume without loss of generality $-\frac{\pi}{2} \leq r(x) \leq \frac{\pi}{2}$ for $x \geq d_{m_0,m_1}^+$ and $\lim_{x\to\infty} r(x) = \frac{\pi}{2}$. Thus there exists $x_1 > d_{m_0,m_1}^+$ such that $0 < r(x) < \frac{\pi}{2}$ for $x > x_1$. By the theorem of Picard-Lindelöf there exists $x_2 \geq x_1$ with $r'(x_2) \neq 0$.

If $r'(x_2) < 0$ the (m_0, m_1)-ODE together with $0 < r(x) < \frac{\pi}{2}$ for $x > x_1$ imply $r''(x) < 0$ for all $x \geq x_2$. Consequently, $r'(x_2) < 0$. By a similar argument, this in turn implies $r''(x) < 0$ for all $x \geq x_2$ and thus $r'(x) < 0$ for $x \geq x_2$. Since $0 < r(x_2) < \frac{\pi}{2}$ this contradicts $\lim_{x\to\infty} r(x) = \frac{\pi}{2}$. If $r'(x_2) > 0$ then we have $r'(x) \geq 0$ for all $x \geq x_2$, otherwise we obtain a contradiction by the same argument as above. Setting $x_0 = x_2$ establishes the claim. $\qquad\square$

3.3.2. *Behavior for large negative x.* We introduce $V_{m_0,m_1}^r : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{1}{2}r'(x)^2 - \beta_{m_0,m_1}(x)\cos^2 r(x)$, which turns out to be a Lyapunov function. The proof of the next lemma is omitted since it can be proved in analogy to the corresponding results of Subsection 3.3.

Lemma 3.9: *Either V_{m_0,m_1}^r decreases strictly on $(-\infty, Z_{m_0,m_1}^\alpha]$ or $V_{m_0,m_1}^r \equiv 0$. In any case $\lim_{x\to-\infty} V_{m_0,m_1}^r(x) \in \mathbb{R} \cup \{\infty\}$ exists. If r satisfies $\lim_{x\to-\infty} r(x) = k\pi$, $k \in \mathbb{Z}$, we have $|r'(x)| \leq \sqrt{m_0}$ for $x \leq Z_{m_0,m_1}^\alpha$.*

In terms of the function ϕ defined by $\phi(x) = r(-x) - \frac{\pi}{2}$ the (m_0, m_1)-ODE becomes

$$(3.2) \qquad \phi''(x) - \alpha_{m_1,m_0}(x)\phi'(x) + \beta_{m_1,m_0}(x)\sin 2\phi(x) = 0,$$

which is the (m_1, m_0)-ODE for ϕ. The next lemma yields restrictions for the first derivative of a solution ϕ of the (m_1, m_0)-BVP.

Lemma 3.10: *Let ϕ be a solution of the (m_1, m_0)-ODE.*

(i) *If* $\phi'(x_0) \geq q_{m_1,m_0}(x_0)$ *for a point* $x_0 > Z^\alpha_{m_1,m_0}$ *then* $\lim_{x\to\infty} \phi(x) = \infty$.

(ii) *If* $\phi'(x_0) \leq -q_{m_1,m_0}(x_0)$ *for a point* $x_0 > Z^\alpha_{m_1,m_0}$ *then* $\lim_{x\to\infty} \phi(x) = -\infty$.

Proof. We assume that there exists a point $x_0 > Z^\alpha_{m_1,m_0}$ such that $\phi'(x_0) \geq q_{m_1,m_0}(x_0)$. By ODE (3.2) the inequality $\phi''(x) > 0$ with $x > Z^\alpha_{m_1,m_0}$ is equivalent to

$$\phi'(x) > q_{m_1,m_0}(x) \sin 2\phi(x).$$

Since $x_0 > Z^\alpha_{m_1,m_0}$ we have $q_{m_1,m_0}(x_0) > 0$. Consequently, $\phi'(x_0) > q_{m_1,m_0}(x_0)$ implies $\phi''(x_0) > 0$. Next we assume that there exists a point $x_1 > x_0$ such that $\phi''(x) > 0$ for all $x \in [x_0, x_1)$ and $\phi''(x_1) = 0$. Since the function q_{m_1,m_0} is strictly decreasing on the interval $(Z^\alpha_{m_1,m_0}, \infty)$, we obtain $\phi'(x) > \phi'(x_0) > q_{m_1,m_0}(x_0) > q_{m_1,m_0}(x)$ for $x \in (x_0, x_1)$. Due to the continuity of the functions ϕ' and q we get $\phi'(x_1) > q_{m_1,m_0}(x_1)$. Since $q_{m_1,m_0}(x_1) > 0$ we have $\phi''(x_1) > 0$, contradicting our assumption. Consequently, we have $\phi''(x) > 0$ for all $x \geq x_0$. This in turn yields $\phi'(x) \geq \phi'(x_0) \geq q_{m_1,m_0}(x_0) > 0$ for all $x \geq x_0$, which establishes the first claim. The second statement is obtained by the first by considering $-\phi$ instead of ϕ. $\qquad\square$

Theorem 3.11: *For* $m_0 \geq 2$ *there exists* $d^-_{m_0,m_1} \in \mathbb{R}$ *such that for each solution* r *of the* (m_0, m_1)-*ODE with* $\lim_{x\to-\infty} r(x) = 0$ *either* $0 \leq r(x) \leq \pi$ *or* $-\pi \leq r(x) \leq 0$, *for all* $x \leq d^-_{m_0,m_1}$.

Proof. First we prove that for each solution ϕ of the ODE (3.2) there exist $\ell_0 \in \mathbb{Z}$ and $C_{m_0,m_1} \in \mathbb{R}$ such that $(2\ell_0 + 1)\frac{\pi}{2} \leq \phi(x) \leq (2\ell_0 + 3)\frac{\pi}{2}$ for all $x \geq C_{m_0,m_1}$, or $\lim_{x\to\infty} \phi(x) = \pm\infty$. Since the proof is similar to that of Lemma 3.7 some details are omitted. Again it is sufficient to deal with the case $\ell_0 = -1$.

Set $C_{m_0,m_1} = \max \left\{ x \in \mathbb{R} \mid 2\beta_{m_1,m_0}(x) = q_{m_1,m_0}(x)^2 \right\}$ and assume that there exists a $x_0 \geq C_{m_0,m_1}$ such that $\phi(x_0) = \frac{\pi}{2}$. Without loss of generality we can assume $\phi'(x_0) > 0$. Then there exists a point $x_1 > x_0$ with $\phi(x_1) = \pi$ and $\phi'(x_1) \geq 0$.

Since $W^\phi_{m_1,m_0}$ is increasing for $x \geq Z^\alpha_{m_1,m_0}$, we obtain

$$W^\phi_{m_1,m_0}(x_1) - W^\phi_{m_1,m_0}(x_0) = \tfrac{1}{2}(\phi'(x_1)^2 - \phi'(x_0)^2) - \beta_{m_1,m_0}(x_0) \geq 0$$

and thus we get $\phi'(x_1) \geq q_{m_1,m_0}(x_0)$, where we used $x_0 \geq C_{m_0,m_1}$ and $\phi'(x_1) \geq 0$. Since q_{m_1,m_0} is strictly decreasing on the interval $(Z^\alpha_{m_1,m_0}, \infty)$ and $x_1 > x_0 \geq C_{m_0,m_1} \geq Z^\alpha_{m_1,m_0}$, we obtain $\phi'(x_1) > q_{m_1,m_0}(x_1)$. Hence Lemma 3.10 implies $\lim_{x\to\infty} \phi(x) = \infty$.

Plugging in $\phi(x) = r(-x) - \frac{\pi}{2}$ implies that there either exists an integer $\ell_0 \in \mathbb{Z}$ such that $\ell_0 \pi \leq r(x) \leq (\ell_0 + 1)\pi$ for all $x \leq -C_{m_0,m_1}$ or $\lim_{x\to-\infty} r(x) = \pm\infty$. Since r satisfies $\lim_{x\to-\infty} r(x) = 0$ we have $\ell_0 \in \{-1, 0\}$. Setting $d^-_{m_0,m_1} = -C_{m_0,m_1}$ establishes the claim. $\qquad\square$

4. The cases $2 \leq m_0 \leq 5$

In the present section we prove Theorem A, i.e., we show that for $2 \leq m_0 \leq 5$ there exit infinitely many solutions of the (m_0, m_1)-BVP and thus infinitely many harmonic maps between the spheres $\mathbb{S}^{m_0+m_1+1}$. For the construction of the solutions we use a shooting method at the singular point $t = 0$.

For $v \in \mathbb{R}$ let r_v be as in Lemma 3.2 and set $\varphi_v := r_v - \frac{\pi}{2}$. We introduce the *nodal number* $\mathfrak{N}(r_v)$ *of* r_v as the number of intersection points of r_v with $\frac{\pi}{2}$. In other words, $\mathfrak{N}(r_v)$ denotes the number of zeros of φ_v. The function $r_1(x) = \arctan(\exp x)$ solves the (m_0, m_1)-BVP with $\mathfrak{N}(r_1) = 0$. The next lemma ensures that for $2 \leq m_0 \leq 5$ we cannot increase v arbitrarily without increasing the nodal number of r_v. Since the proof of Lemma 4.2 in [11] does not depend on the sign of λ_2, we obtain the following result.

Lemma 4.1 (see [11])**:** *Let* $2 \leq m_0 \leq 5$. *For each* $k \in \mathbb{N}$ *there exists* $c(k) > 0$ *such that* $\mathfrak{N}(r_v) \geq k$ *whenever* $v > c(k)$.

We now prove Theorem A. The idea can be explained with the help of the following pictures. In the left picture a solution of the initial value problem with nodal number
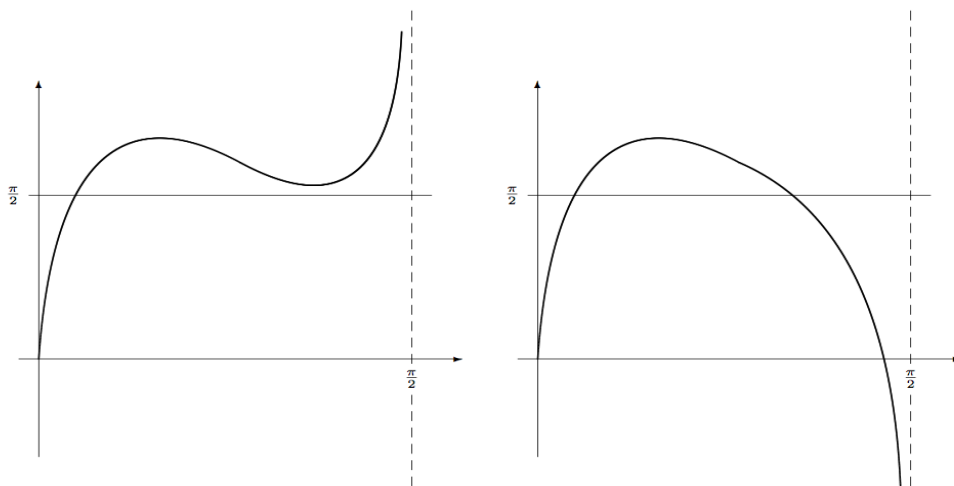
1 is sketched. By Lemma 4.1 the nodal number increases if we increases the initial velocity large enough. We will prove that the nodal number can increases always by 1. So there exists an initial velocity for which the nodal number is given by 2; this situation is sketched in the right picture. Using an intermediate value theorem we prove that this implies the existence of a solution of the (m_0, m_1)-BVP with nodal number 1. Afterwards we proceed inductively to prove the claim.

Theorem 4.2: *Let* $2 \leq m_0 \leq 5$. *For each* $k \in \mathbb{N}_0$ *there exists a solution* r_v *of the* (m_0, m_1)-*BVP with* $\mathfrak{N}(r_v) = k$.

Proof. The function $r_1(x) = \arctan(\exp(x))$ solves the (m_0, m_1)-BVP with $\mathfrak{N}(r_1) = 0$. Consequently, $v_0 = \sup \{v \mid \mathfrak{N}(r_v) = 0\}$ is well-defined and Lemma 4.1 implies $v_0 < \infty$.

We prove by contradiction that $\mathfrak{N}(r_{v_0}) = 0$. Assume that there exists $x_0 \in \mathbb{R}$ with $r_{v_0}(x_0) = \frac{\pi}{2}$. By Lemma 3.2 we have $r'_{v_0}(x_0) \neq 0$. Consequently, φ_{v_0} has opposite signs in the intervals $(-\infty, x_0)$ and (x_0, ∞), respectively. Since r_v depends continuously on v there exists a sequence $(c_i)_{i \in \mathbb{N}}$ with $c_i < v_0$, $\lim_{i \to \infty} c_i = v_0$ and $\mathfrak{N}(r_{c_i}) = 0$. Thus each of the functions φ_{c_i} has a different sign than φ_{v_0} on the interval (x_0, ∞). This contradicts the fact that φ_v depends continuously on v. Consequently, $\mathfrak{N}(r_{v_0}) = 0$.

By Lemma 3.2 there cannot exist a point $x_0 \in \mathbb{R}$ such that $r_v(x_0) = \frac{\pi}{2}$ and $r_v'(x_0) = 0$. Since r_v depends continuously on v, an additional node can only arise at infinity, i.e., there exists $\epsilon > 0$ such that φ_v has at least one zero $z_1(v)$ for each $v \in (v_0, v_0 + \epsilon)$ and $\lim_{v \searrow v_0} z_1(v) = \infty$. Lemma 3.7 implies that we can choose $\epsilon > 0$ such that φ_v has exactly one zero $z_1(v)$ for each $v \in (v_0, v_0 + \epsilon)$.

There exists $\widetilde{v}$ such that $z_1(\widetilde{v}) < d_{m_0,m_1}^+$ for all $v > \widetilde{v}$: if $z_1(v) > d_{m_0,m_1}^+$ for all $v > v_0$ then Lemma 3.7 implies $\lim_{x \to \infty} \varphi_v(x) = \infty$ for all $v > v_0$. However, Lemma 4.1 ensures that for $2 \leq m_0 \leq 5$ we cannot increase v arbitrarily without increasing the number of zeros of φ_v. Since additional nodes have to arise at infinity such an $\widetilde{v}$ exists. By the preceding considerations $v_1 = \sup \{v \,|\, \mathfrak{N}(r_v) = 1\}$ is well-defined and $v_1 > v_0$. Furthermore, Lemma 4.1 implies $v_1 < \infty$.

We can now proceed inductively, i.e., $v_k = \sup \{v \,|\, \mathfrak{N}(r_v) = k\}$ is well-defined, satisfies $v_k > v_{k-1}$ and is finite for each $k \in \mathbb{N}$. Analogously to the considerations for v_1 we prove that φ_{v_k} has exactly k zeros and that there exists $\epsilon_k > 0$ such that each φ_v, $v \in (v_k, v_k + \epsilon_k)$, has exactly $k + 1$ zeros.

Finally, we prove that there exists $\ell_0 \in \mathbb{Z}$ such that $\lim_{x \to \infty} r_{v_i}(x) = \ell_0 \pi + \frac{\pi}{2}$ and thus each r_{v_i}, $i \in \mathbb{N}$, is a solution of the (m_0, m_1)-BVP: if no such $\ell_0 \in \mathbb{Z}$ exists, Lemma 3.5 implies $\lim_{x \to \infty} r_{v_i}(x) = \pm\infty$. We may assume without loss of generality $\lim_{x \to \infty} r_{v_i}(x) = -\infty$. Recall $\mathfrak{N}(r_{v_i}) = i$ and $\mathfrak{N}(r_v) = i + 1$ for $v \in (v_i, v_i + \epsilon_i)$. Consequently, for each $v \in (v_i, v_i + \epsilon_i)$ there exists $x_v \in \mathbb{R}$ such that $\varphi_v(x_v) = 0$ and $\varphi_v(x) > 0$ for all $x > x_v$. Since φ_v depends continuously on v we get $\lim_{v \to v_i} x_v = \infty$. Hence there exists $\hat{\epsilon}_i \leq \epsilon_i$ such that $x_v > d_{m_0,m_1}^+$ for all $v \in (v_i, v_i + \hat{\epsilon}_i)$. Lemma 3.7 thus implies $\lim_{x \to \infty} \varphi_v(x) = \infty$ for $v \in (v_i, v_i + \hat{\epsilon}_i)$.

On the other hand, the fact that φ_v depends continuously on v implies that for each $v \in (v_i, v_i + \hat{\epsilon}_i)$ there exist $k_0 \in \mathbb{Z}$ and $x_{k_0} > d_{m_0,m_1}^+$ such that $\varphi_v(x_{k_0}) = (2k_0 + 1)\frac{\pi}{2}$ and $\varphi_v'(x_{k_0}) < 0$. Lemma 3.7 thus implies $\lim_{x \to -\infty} \varphi_v(x) = -\infty$, which contradicts the results of the preceding paragraph. Consequently, there exists $\ell_0 \in \mathbb{Z}$ such that $\lim_{x \to \infty} r_{v_i}(x) = \ell_0 \pi + \frac{\pi}{2}$ and thus each r_{v_i}, $i \in \mathbb{N}$, is a solution of the (m_0, m_1)-BVP. $\square$

Remark 4.3:   (a) If r is a solution of the (m_0, m_1)-BVP, so is $-r$. Consequently, for each solution of the (m_0, m_1)-BVP with $v > 0$ there exists a second solution which is obtained from the original solution by reflection on the x-axis.

(b) The present approach allows to treat the BVP considered by Bizon and Chmaj in [3] (see Subsection 2.3) and the (m_0, m_1)-BVP in a unified way: the ODE for the Hopf construction reduces to ODE investigated by Bizon and Chmaj if we choose $p = q = \mu = \lambda = m$ and $\alpha = r$. Following [3] we introduce the function $W : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{1}{2}h'(x)^2 + \frac{m}{2}\sin^2 h(x)$, where $h : \mathbb{R} \to \mathbb{R}$ is given by $h = r - \frac{\pi}{2}$. Similar as in the present section we may now prove the existence of an infinite family of harmonic self-maps of $\mathbb{S}^{m+1}$ for $2 \leq m \leq 5$.

5. The cases $m_0 \geq 6$

In this section we prove Theorem C, i.e., we show that for $m_0 \geq 6$, the nodal number of the solutions r_v, $v \in \mathbb{R}$, of the (m_0, m_1)-ODE is bounded. This explains why for $m_0 \geq 6$ we cannot proceed analogous to the cases $2 \leq m_0 \leq 5$ in order to construct

infinitely many solutions of the (m_0, m_1)-BVP. Note that this is not a statement about nonexistence, since there might be other ways to construct (infinitely many) solutions of the (m_0, m_1)-BVP for $m_0 \geq 6$. The following question remains open.

Question 1: If $m_0 \geq 6$, do there exist more solutions of the (m_0, m_1)-BVP than $r(t) = \pm t$? If the answer is affirmative, how do you prove there existence?

Strategy for proving boundedness of the nodal number. Recall that the function $\varphi_v(x) = r_v(-x) - \frac{\pi}{2}$ satisfies the (m_1, m_0)-ODE. We introduce the continuous function $\theta_v : \mathrm{I} \to \mathbb{R}, x \mapsto \arctan \frac{\varphi_v'(x)}{\varphi_v(x)}$, where $\mathrm{I} = [-d_{m_0, m_1}^+, \infty)$. By Lemma 3.2 the limit $\theta_v(\infty) = \lim_{x \to \infty} \theta_v(x)$ exists and is finite and thus $\Omega_v = -\frac{1}{\pi}(\theta_v(\infty) - \theta_v(-d_{m_0, m_1}^+))$ is well-defined. Ω_v will henceforth be referred to as winding number of φ_v. Lemma 3.7 implies that the difference between $\mathfrak{N}(r_v)$ and $\lfloor \Omega_v \rfloor$ is at most one. Hence to prove that $\mathfrak{N}(r_v)$, $v \in \mathbb{R}$, is bounded from above by an integer which depends on m_0 and m_1 only, it is sufficient to show the same for $\lfloor \Omega_v \rfloor$ instead. In order to prove this, we consider the linearized (m_1, m_0)-ODE. Similarly as above we associate a winding number Ω_L to this linearized differential equation and prove that Ω_L is larger than $\lfloor \Omega_v \rfloor$. Finally, we show that Ω_L is bounded from above by a constant which only depends on m_0 and m_1 and thus establish the claim.

Proof that the nodal number is bounded. The following two lemma are the main ingredients for the proof of Theorem C. As indicated above, we start by considering the linearized (m_1, m_0)-ODE and prove that its winding number is an upper bound for the winding number of the (m_1, m_0)-ODE. Below let φ_L be a solution of the linear differential equation

(5.1) $$\varphi_L''(x) - \alpha_{m_1, m_0}(x)\varphi_L'(x) + m_0 \varphi_L(x) = 0.$$

Furthermore, we introduce the continuous function $\theta_L : \mathrm{I} \to \mathbb{R}$ by $\theta_L(x) = \arctan \frac{\varphi_L'(x)}{\varphi_L(x)}$.

Lemma 5.1: *Let $m_0 \geq 6$.*
  (i) *For $v \in \mathbb{R}$ with $\theta_v(-d_{m_0, m_1}^+) \geq \theta_L(-d_{m_0, m_1}^+)$ we have $\theta_v \geq \theta_L$ on I.*
  (ii) *The limit $\lim_{x \to \infty} \theta_L(x)$ exists and is finite.*

Proof. We start by proving the first part. Its proof is similar to that of Lemma 6 in [3]. By (3.2) we obtain

$$\theta_v'(x) = -\sin^2 \theta_v(x) + \tfrac{1}{2}\alpha_{m_1, m_0}(x)\sin 2\theta_v(x) - 2\beta_{m_1, m_0}(x)\cos^2 \theta_v(x)\tfrac{\sin 2\varphi_v(x)}{2\varphi_v(x)}.$$

Similarly, (5.1) yields

(5.2) $$\theta_L'(x) = -\sin^2 \theta_L(x) + \tfrac{1}{2}\alpha_{m_1, m_0}(x)\sin 2\theta_L(x) - m_0 \cos^2 \theta_L(x).$$

Consequently, $u := \theta_v - \theta_L$ satisfies $u'(x) = s_1(x)u(x) + s_2(x)$, with

$$s_1(x) = \tfrac{1}{2}\alpha_{m_1, m_0}(x)\tfrac{\sin 2\theta_v(x) - \sin 2\theta_L(x)}{\theta_v(x) - \theta_L(x)} + (m_0 - 1)\tfrac{\sin^2 \theta_v(x) - \sin^2 \theta_L(x)}{\theta_v(x) - \theta_L(x)},$$
$$s_2(x) = \big(m_0 - 2\beta_{m_1, m_0}(x)\tfrac{\sin 2\varphi_v(x)}{2\varphi_v(x)}\big)\cos^2 \theta_v(x).$$

Variation of parameters yields

$$u(x) = \exp(F(x)) \left(\int_{-d_{m_0,m_1}^+}^{x} s_2(\xi) \exp(-F(\xi)) d\xi + c \right),$$

where $F(x) = \int_{-d_{m_0,m_1}^+}^{x} s_1(\xi) d\xi$ and $c \in \mathbb{R}$. The condition $u(-d_{m_0,m_1}^+) \geq 0$ implies $c \geq 0$. Since $s_2 \geq 0$ we get $u(x) \geq 0$ for all $x \geq -d_{m_0,m_1}^+$, completing the proof of the first part.

Below we prove the second claim of this lemma. Introduce the function $h : \mathbb{R}^2 \to \mathbb{R}$ by $h(t, x) = -\sin^2 t + \frac{1}{2}\alpha_{m_1,m_0}(x)\sin 2t - m_0 \cos^2 t$. For $x_0 := \operatorname{artanh}(\frac{4\sqrt{m_0}+m_1-m_0}{m_0+m_1-2})$ and $t_0 = -\arccos(-\frac{1}{\sqrt{1+m_0}})$ we have $h(t_0, x_0) = 0$. Since $\sin 2t_0 > 0$, $\alpha_{m_1,m_0}(x_0) > 0$ and α_{m_1,m_0} is strictly increasing, we get

(5.3) $h(t_0 + k\pi, x) = h(t_0, x) > h(t_0, x_0) = 0$ for all $x > x_0, k \in \mathbb{Z}$.

Either $\theta_L'(x) \leq 0$ for all $x \geq x_0$ or there exists $x_1 \geq x_0$ such that $\theta_L'(x_1) > 0$. In the first case (5.2) and (5.3) imply that θ_L is bounded. Hence the limit $\lim_{x \to \infty} \theta_L(x)$ exists and is finite, whence the claim. In the second case either (a) $\theta_L'(x) \geq 0$ for all $x \geq x_1$, or (b) there exists a point $x_4 > x_1$ such that $\theta_L'(x_4) < 0$.

Since $h(\frac{\pi}{2} + k\pi, x) < 0$ for $k \in$ and $x \geq x_0$, differential equation (5.2) implies that θ_L is bounded in case (a). Consequently, $\lim_{x \to \infty} \theta_L(x)$ exists and is finite. On the other hand (b) cannot occur: since θ_L' is continuous there exist $x_2, x_3 \in (x_1, x_4)$ with $x_2 < x_3$ such that $\theta_L'(x_2) > 0$, $\theta_L'(x_3) < 0$ and $\theta_L(x_2) = \theta_L(x_3)$. Using differential equation (5.2) we find that $\theta_L'(x_2) > 0$ implies $\sin 2\theta_L(x_2) > 0$. Since $\alpha_{m_1,m_0}(x)$ is positive for $x \geq x_0$ and increasing we get

$$0 < \theta_L'(x_2) = -\sin^2 \theta_L(x_3) + \tfrac{1}{2}\alpha_{m_1,m_0}(x_2)\sin 2\theta_L(x_3) - m_0 \cos^2 \theta_L(x_3) \leq \theta_L'(x_3) < 0,$$

a contradiction. $\qquad\qquad\square$

The preceding lemma implies that $\Omega_L = -\frac{1}{\pi}(\theta_L(\infty) - \theta_L(-d_{m_0,m_1}^+))$ is well-defined.

Lemma 5.2: *Let $m_0 \geq 6$.*

(i) *For $v \in \mathbb{R}$ with $\theta_v(-d_{m_0,m_1}^+) = \theta_L(-d_{m_0,m_1}^+)$ we have $\Omega_v \leq \Omega_L$.*

(ii) *Ω_L is bounded from above by a constant which only depends on m_0 and m_1.*

Proof. We start by proving (i). The first part of Lemma 5.1 implies $\theta_v - \theta_L \geq 0$ on I which is equivalent to the inequality

$$-\tfrac{1}{\pi}(\theta_v(x) - \theta_v(-d_{m_0,m_1}^+)) \leq -\tfrac{1}{\pi}(\theta_L(x) - \theta_L(-d_{m_0,m_1}^+))$$

for all $x \geq -d_{m_0,m_1}^+$. Hence we get $\Omega_v \leq \Omega_L$ which establishes the claim.

Next we prove (ii). The differential equation (5.2) yields $l_1 \leq \theta_L'(x) \leq l_2$ where $l_1 = -\frac{1}{2}(2m_0 + m_1 + 1)$ and $l_2 = \frac{1}{2}(m_0 - 1)$. Hence

(5.4) $(x + d_{m_0,m_1}^+)l_1 \leq \theta_L(x) - \theta_L(-d_{m_0,m_1}^+) \leq (x + d_{m_0,m_1}^+)l_2$

for $x \in \mathbb{R}$. Let $x_0 \in \mathbb{R}$ be as in the proof of Lemma 5.1. Then either
(a) $\theta_L'(x) \leq 0$ for all $x \geq x_0$, or
(b) there exists $x_1 > x_0$ such that $\theta_L'(x) \geq 0$ for all $x \geq x_1$ and $\theta_L'(x) \leq 0$ for $x_0 \leq x < x_1$.

In case (a) the proof of Lemma 5.1 yields $|\theta_L(x_0) - \theta_L(x)| \leq \pi$ for $x \geq x_0$. Combining this inequality with (7.7) applied to $x = x_0$ we get

$$(x_0 + d^+_{m_0,m_1})l_1 - \pi \leq \theta_L(x) - \theta_L(-d^+_{m_0,m_1}) \leq (x_0 + d^+_{m_0,m_1})l_2 + \pi \ \text{ for all } \ x \geq x_0.$$

Taking the limit as x approaches infinity we obtain

$$-1 - \tfrac{1}{\pi}(x_0 + d^+_{m_0,m_1})l_2 \leq \Omega_L \leq 1 - \tfrac{1}{\pi}(x_0 + d^+_{m_0,m_1})l_1.$$

Since the right hand side of the previous inequality obviously depends on m_0 and m_1 only, this establishes the claim in case (a).

In case (b) the proof of Lemma 5.1 implies $|\theta_L(x) - \theta_L(x_0)| \leq \pi$ for $x_0 \leq x < x_1$. Furthermore, since $h(\frac{\pi}{2} + k\pi, x) < 0$ we have $|\theta_L(x) - \theta_L(x_1)| \leq \pi$ for $x \geq x_1$. Consequently, $|\theta_L(x) - \theta_L(x_0)| \leq 2\pi$ for all $x \geq x_0$. Now proceed as in case (a). $\qquad \square$

Finally, we are now ready to prove Theorem C.

Theorem 5.3: *For $m_0 \geq 6$ the nodal number of r_v, $v \in \mathbb{R}$, is bounded from above by a constant which only depends on m_0 and m_1.*

Proof. Choose $\varphi_L(-d^+_{m_0,m_1})$ and $\varphi'_L(-d^+_{m_0,m_1})$ such that $\theta_v(-d^+_{m_0,m_1}) = \theta_L(-d^+_{m_0,m_1})$. Consequently, $\mathfrak{N}(r_v) \leq \lfloor \Omega_v \rfloor + 1$ and Lemma 5.2 imply

$$\mathfrak{N}(r_v) \leq \lfloor \Omega_v \rfloor + 1 \leq \Omega_L + 1 \leq N_0 + 1,$$

which establishes the claim. $\qquad \square$

Remark 5.4: The estimates used in this section are not optimal. For example the bound on θ_L in the proof of Lemma 5.2 can easily be improved.

6. Behaviour for large initial velocities

Throughout this section let $m_1 \geq m_0 \geq 2$, $r_v : \mathbb{R} \to \mathbb{R}$ as in Lemma 3.2 and set $\varphi_v = r_v - \frac{\pi}{2}$. We show that the solutions r_v of the (m_0, m_1)-BVP converge to a limiting configuration as v goes to infinity, namely, for large enough initial velocities r_v becomes arbitrarily close to $\frac{\pi}{2}$ on each open interval in $(-\infty, \infty)$. As a byproduct we prove that for $2 \leq m \leq 5$ there are infinitely many solutions of the (m,m)-BVP with nodal number zero.

The following two lemma are used in the proof of Theorem D. In the next lemma we show that for every interval of the form $[x_0, d^-_{m_0,m_1}]$, the energy $V^{r_v}_{m_0,m_1}$ becomes arbitrarily small on this interval if we chose the velocity v to be "large enough".

Lemma 6.1: *For $\epsilon > 0$ and $x_0 \leq d^-_{m_0,m_1}$ there exists $v_0 > 0$ such that $V^{r_v}_{m_0,m_1}(x) < \epsilon$ for $x_0 \leq x \leq d^-_{m_0,m_1}$ and $v \geq v_0$.*

Proof. From $\lim_{x \to -\infty} r'_v(x) = 0$ we have that there exists $x_1 \leq d^-_{m_0,m_1}$ such that $r'_v(x)^2 < \epsilon$ for $x \leq x_1$. Furthermore, by the proof of Lemma 3.3 in [11] we get

$$(6.1) \qquad \qquad \lim_{v \to \infty} \varphi_v(x - \log v) = \psi(x)$$

for all $x \in \mathbb{R}$, where $\psi : \mathbb{R} \to \mathbb{R}$ denotes the unique solution of

$$\psi''(x) + (m_0 - 1)\psi'(x) + \tfrac{1}{2}m_0 \sin 2\psi(x) = 0,$$

satisfying $\psi(x) \simeq -\frac{\pi}{2} + \exp(x)$ as $x \to -\infty$. From [11] we further have $\lim_{x\to\infty} \psi(x) = 0$. Consequently, for a given $\epsilon_0 > 0$ there exists $x_2 \in \mathbb{R}$ such that $2|\psi(x_2)| < \epsilon_0$. By (6.1) there exists an $v_0 \in \mathbb{R}$ such that $|\varphi_v(x_2 - \log v)| < \epsilon_0$ for all $v \geq v_0$. Since β_{m_0,m_1} is bounded, we can choose $\epsilon_0 > 0$ so small that

$$2|\beta_{m_0,m_1}(x_2 - \log v)\sin^2 \varphi_v(x_2 - \log v)| < \epsilon$$

for all $v \geq v_0$. We may assume that v_0 is so large that $x_2 - \log v_0 \leq \min(x_0, x_1)$. Thus we get $V_{m_0,m_1}^{r_v}(x_2 - \log v) < \epsilon$ for $v \geq v_0$. Since $V_{m_0,m_1}^{r_v}$ decreases strictly on the interval $(-\infty, d_{m_0,m_1}^-]$ this implies the claim. $\qquad\square$

Following the proof of Lemma 4 in [3] we show that $(\varphi_v(x), \varphi_v'(x))$ stays close to zero for bounded $x \geq d_{m_0,m_1}^-$ provided that v is chosen large enough. As in [3] we introduce the distance function $\rho_v : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sqrt{\varphi_v(x)^2 + \varphi_v'(x)^2}$, which satisfies $\rho_v > 0$ by Lemma 3.2.

Lemma 6.2: *For any $x_0, x_1 \in \mathbb{R}$ with $x_0 \leq x_1$ and $\eta > 0$, there exists $v_0 \in \mathbb{R}$ such that $v \geq v_0$ implies $\rho_v(x) < \eta$ for $x_0 \leq x \leq x_1$.*

Proof. The (m_0, m_1)-ODE implies

$$\rho_v(x)\rho_v'(x) = \varphi_v(x)\varphi_v'(x) + \alpha_{m_0,m_1}(x)\varphi_v'(x)^2 + 2\beta_{m_0,m_1}(x)\frac{\sin 2\varphi_v(x)}{2\varphi_v(x)}\varphi_v(x)\varphi_v'(x)$$
$$\leq (m_1 + 1)|\varphi_v(x)\varphi_v'(x)| + (m_1 - 1)\varphi_v'(x)^2 \leq c\rho_v(x)^2,$$

where we use $\varphi_v'(x)^2 \leq \rho_v(x)^2$, $2|\varphi_v(x)\varphi_v'(x)| \leq \rho_v(x)^2$ and set $c = \frac{1}{2}(3m_1 - 1)$. Thus $\frac{\rho_v'(x)}{\rho_v(x)} \leq c$. Integrating this inequality from a given $T_- \leq \min(x_0, d_{m_0,m_1}^-)$ to a point $x \geq T_-$ yields

$$(6.2) \qquad\qquad \rho_v(x) \leq \exp(c(x - T_-))\rho_v(T_-).$$

Lemma 6.1 guarantees for every $\epsilon > 0$ the existence of a velocity $v_1 > 0$ such that $V_{m_0,m_1}^{r_v}(T_-) < \epsilon$ for all $v \geq v_1$. Since for $x \leq d_{m_0,m_1}^-$ both summands in the definition of $V_{m_0,m_1}^r(x)$ are positive we get

$$|\varphi_v'(T_-)| < \sqrt{2\epsilon} \quad \text{and} \quad \sin^2 \varphi_v(T_-) < \frac{\epsilon}{|\beta_{m_0,m_1}(T_-)|}$$

for all $v \geq v_1$. Since $v > 0$ Theorem 3.11 implies $\rho_v(T_-)$ becomes arbitrarily small if ϵ converges to zero. Consequently, for any $T_+ \geq \max(x_1, d_{m_0,m_1}^-)$ and $\eta > 0$ there exists a velocity $v_2 > 0$ such that

$$\rho_v(T_-) < \exp(-c(T_+ - T_-))\eta$$

for all $v \geq v_2$. Substituting this into (6.2) yields $\rho_v(x) < \eta$ for $T_- \leq x \leq T_+$ and $v \geq v_0 := \max(v_1, v_2)$, whence the claim. $\qquad\square$

We are now ready to prove Theorem D, i.e., that for each $x_0 \in \mathbb{R}$ there exists a velocity v_0 such that r_v becomes arbitrarily close to $\frac{\pi}{2}$ on the interval (x_0, ∞).

Theorem 6.3: *Let ρ_v be the distance function associated to a solution r_v of the (m_0, m_1)-BVP. For $\epsilon > 0$ and $x_0 \in \mathbb{R}$ there exists $v_0 \in \mathbb{R}$ such that $\rho_v(x) < \epsilon$ for $x \geq x_0$, $v \geq v_0$.*

Proof. Since $\lim_{x\to\infty} \alpha_{m_0,m_1}(x) = m_1 - 1$ and α_{m_0,m_1} is strictly increasing there exists $x_0 \in \mathbb{R}$ such that $\alpha_{m_0,m_1}(x) \geq \frac{m_1-1}{2}$ for $x \geq x_0$. Moreover, since $\lim_{x\to\infty} \beta_{m_0,m_1}(x) = \frac{m_1}{2}$, for each $0 < \lambda < 1$ there exists $x_1 \in \mathbb{R}$ such that $\beta_{m_0,m_1}(x) \geq \lambda \frac{m_1}{2}$ for $x \geq x_1$. Set $T = \max(x_0, x_1, d^+_{m_0,m_1})$. Lemma 6.2 implies the existence of $v_0 \in \mathbb{R}$ such that $\rho_v(x) < \epsilon$ for $d^-_{m_0,m_1} \leq x \leq T$ and $v \geq v_0$. In particular $|\varphi_v(x)| < \epsilon$ for $v \geq v_0$ and $d^-_{m_0,m_1} \leq x \leq T$.

Let $\mu \in (0,1)$ and $\lambda > \mu$ be given. Since $W^{r_v}_{m_0,m_1}(T) \geq \lambda \frac{m_1}{2} \sin^2 r_v(T)$ we can assume that $0 < \epsilon < \frac{\pi}{2}$ is so small that $|\varphi_v(T)| < \epsilon$ for $v \geq v_0$ implies $W^{r_v}_{m_0,m_1}(T) \geq \mu \frac{m_1}{2}$ for all $v \geq v_0$. Since $\lim_{x\to\infty} W^{r_v}_{m_0,m_1}(x) = \frac{m_1}{2}$ and $W^{r_v}_{m_0,m_1}$ increases strictly on the interval $[T, \infty)$, we get

$$(6.3) \qquad 0 \leq W^{r_v}_{m_0,m_1}(x) - W^{r_v}_{m_0,m_1}(T) \leq (1-\mu)\frac{m_1}{2} \quad \text{for all } x \geq T, v \geq v_0.$$

Let $\delta \in \mathbb{R}$ with $|\delta| < \frac{\pi}{2}$ be given. Furthermore, consider a fixed μ with

$$\max(\tfrac{1}{2}(1 + \sin^2 \delta), 1 - \tfrac{1}{4}(m_1 - 1)(1 - \sin^2 \delta)\epsilon_0)) < \mu < 1.$$

In what follows we assume $r_v(x_3) = k_0\pi + \delta$ for an $k_0 \in \mathbb{Z}$, $x_3 \geq T$ and an $v \geq v_0$. Hence we obtain $W^{r_v}_{m_0,m_1}(x_3) = \frac{1}{2}r'_v(x_3)^2 + \beta_{m_0,m_1}(x_3)\sin^2\delta \geq \frac{1}{2}m_1\mu$ and thus

$$\frac{d}{dx} W^{r_v}_{m_0,m_1}(x_3) \geq \alpha_{m_0,m_1}(x_3)r'_v(x_3)^2 \geq (m_1 - 1)\left(\tfrac{1}{2}m_1\mu - \beta_{m_0,m_1}(x_3)\sin^2\delta\right)$$
$$\geq \tfrac{1}{4}(m_1 - 1)m_1(1 - \sin^2\delta).$$

Lemma 3.4 and Lemma 3.9 imply that the absolute value of the second derivative of $W^{r_v}_{m_0,m_1}$ is bounded. Consequently, there exists an $\epsilon_0 > 0$ which depends only on δ, m_0 and m_1 such that $\frac{d}{dx} W^{r_v}_{m_0,m_1}(x) \geq \tfrac{1}{8}(m_1 - 1)m_1(1 - \sin^2\delta)$ for $x \in [x_2, x_2 + \epsilon_0]$. Thus

$$W^{r_v}_{m_0,m_1}(x_3 + \epsilon_0) - W^{r_v}_{m_0,m_1}(T) \geq W^{r_v}_{m_0,m_1}(x_3 + \epsilon_0) - W^{r_v}_{m_0,m_1}(x_3) \geq \tfrac{m_1}{8}(m_1 - 1)(1 - \sin^2\delta)\epsilon_0.$$

On the other hand inequality (6.3) implies

$$W^{r_v}_{m_0,m_1}(x_3 + \epsilon) - W^{r_v}_{m_0,m_1}(T) \leq (1 - \mu)\tfrac{m_1}{2} < \tfrac{m_1}{8}(m_1 - 1)(1 - \sin^2\delta)\epsilon_0.$$

Hence a point $x_3 \in \mathbb{R}$ with the properties stated above cannot exist.

Since $|\varphi_v(x)| < \epsilon$ for $v \geq v_0$ and $d^-_{m_0,m_1} \leq x \leq T$ we have: for $k_0 = 0$ and $\delta = \frac{\pi}{2} - \epsilon$, there exists an v_0^1 such that $r_v(x) > \frac{\pi}{2} - \epsilon$ for all $v \geq v_0^1$ and $x \geq T$. For $k_0 = 1$ and $\delta = -(\frac{\pi}{2} - \epsilon)$, there exists an v_0^2 such that $r_v(x) < \frac{\pi}{2} + \epsilon$ for all $v \geq v_0^2$ and $x \geq T$. Thus $|\varphi_v(x)| < \epsilon$ for all $v \geq v_1 := \max(v_0^1, v_0^2)$ and $x \geq T$.

Let $0 < \lambda \leq \frac{1}{2}$. By applying the preceding considerations to $\lambda\epsilon$ instead of ϵ, there exist $T \in \mathbb{R}$ and $v_1 \in \mathbb{R}$ such that $|r_v(x) - \frac{\pi}{2}| < \lambda\epsilon$ for all $v \geq v_1$ and $x \geq T$. We may assume that λ and T are chosen such that $(m_1 - 2\beta_{m_0,m_1}(x)\sin^2 r_v(x))^{\frac{1}{2}} < \frac{\epsilon}{2}$ for $v \geq v_1$, $x \geq T$. Since $W^{r_v}_{m_0,m_1}$ is increasing on the interval $[T, \infty)$ and $\lim_{x\to\infty} W^{r_v}_{m_0,m_1}(x) = \frac{m_1}{2}$ for any solution r_v of the (m_0, m_1)-BVP, this implies $|r'_v(x)| < \frac{\epsilon}{2}$ for $v \geq v_1$, $x \geq T$. Consequently, $\rho_v(x) < \epsilon$ for $v \geq v_1$, $x \geq T$. Combining this result with Lemma 6.2 establishes the claim. $\qquad\square$

Below we apply the above theorem in order to prove that for $2 \leq m \leq 5$ there exists an infinite family of harmonic self-maps of $\mathbb{S}^{2m+1}$ with nodal number zero and thereby establish Theorem B.

Theorem 6.4: *Let $m = m_0 = m_1$. For $2 \leq m \leq 5$ there exist an infinite family of harmonic self-maps of $\mathbb{S}^{2m+1}$ with nodal number zero.*

Proof. By Theorem 4.2 there exists a countably infinite family of solutions of the (m,m)-BVP. If we reflect each member of the infinite family on the point $\left(0, \frac{\pi}{4}\right)$, we obtain again an infinite family of solutions of the (m,m)-BVP. Indeed, if r is a solution of the (m,m)-ODE, so are the functions defined by $x \mapsto \frac{(2k+1)\pi}{2} - r(-x)$, $k \in \mathbb{Z}$.

Theorem 6.3 implies that for $\epsilon > 0$ there exists an $v_0 \in \mathbb{R}$ such that $|r_v(x) - \frac{\pi}{2}| < \epsilon$ for all solutions r_v of the (m,m)-BVP with $v \geq v_0$ and $x \geq d_{m,m}^-$. For a solution r_v of the (m,m)-BVP we denote by s_v the solution which we obtain by reflection of r_v on the point $\left(0, \frac{\pi}{4}\right)$. Hence $|s_v(x)| < \epsilon$ for all solutions s_v of the (m,m)-BVP with $v \geq v_0$ and $x \leq -d_{m,m}^- = d_{m,m}^+$. Lemma 3.7 implies $\mathfrak{N}(s_v) = 0$.

The claim follows as soon as we know there exists infinitely many solutions s_v of the (m,m)-BVP with $v \geq v_0$. This is an easy consequence of Theorem 4.2: set $a_k = \inf\{c \mid \mathfrak{N}(r_v) \geq k \text{ whenever } v > c\}$ which is well-defined by Lemma 4.1. Clearly, a_k is an increasing sequence. If $A = \lim_{k \to \infty} a_k < \infty$ then $\mathfrak{N}(r_v) = \infty$ for $v \geq A$. However, Lemma 3.7 implies that each r_v has finite nodal number. Consequently, $\lim_{k \to \infty} a_k = \infty$ and thus the proof of Theorem 4.2 implies that if $2 \leq m_0 \leq 5$ for each $v_0 \in \mathbb{R}$ there exist infinitely many solutions of the (m_0, m_1)-BVP with $v \geq v_0$. $\qquad\square$

Application: infinite families of harmonic self-maps of the special orthogonal group. By Theorem 6.2 in [15] any solution of the (m,m)-BVP yields a harmonic self-map of $SO(m+2)$. Thus Theorems 4.2 and 6.4 imply the following result, Theorem G.

Theorem 6.5: *For each of the special orthogonal groups $SO(4), SO(5), SO(6)$ and $SO(7)$ there exists two infinite families of harmonic self-maps.*

Proof. While the solutions constructed in Theorems 6.4 all have nodal number 0, only one member of the infinite family constructed in Theorem 4.2 has nodal number 0. Thus there are two families of harmonic self-maps of $SO(m)$, $4 \leq m \leq 7$, which have at most one element in common. $\qquad\square$

7. Restrictions on the Brouwer degree

In the first subsection we prove that the Brouwer degree of each solution r of the (m_0, m_1)-BVP with $m_0 \geq 2$ is either ± 1 or ± 3. In the second subsection we show that the Brouwer degree of r_v is given by ± 1 if we chose v "sufficiently large", i.e., for all $m_0, m_1 \in \mathbb{N}$ with $m_0 \leq m_1$ there exists a velocity v_0 such that the Brouwer degree of each solution r_v of the (m_0, m_1)-BVP with $v \geq v_0$ is given by ± 1. Throughout this section we assume that r satisfies the (m_0, m_1)-ODE.

7.1. **Possible Brouwer degrees of the solutions r of the (m_0, m_1)-BVP.** The next lemma provides several estimates which we use in the proof of Theorem E. Introduce the abbreviations $R_{m_0, m_1} = d_{m_0, m_1}^+ - Z_{m_0, m_1}^\alpha$ and $L_{m_0, m_1} = Z_{m_0, m_1}^\alpha - d_{m_0, m_1}^-$.

Lemma 7.1: *For $m_0 \geq 2$ we have*

   (a) $R_{m_0, m_1} \leq artanh(\frac{5}{8m_0 - 3})$ for all $m_1 \geq \max(m_0, 4)$,

(b) $\sqrt{m_0}\, L_{m_0,m_1} \leq \sqrt{m_0}\, artanh(\frac{1}{3m_0-4}) < \frac{\pi}{2}$ for all $m_1 \geq m_0$,

(c) $L_{m_0,m_1} \geq artanh(\frac{1+\sqrt{17}}{16m_0-(17+\sqrt{17})})$ for all $m_1 \geq 3m_0 - 4$.

Proof. Replacing m_1 with a real number we can consider d^+, d^-, Z^α and thus R and L as real valued functions on $\mathbb{N} \times I$ for a suitable interval I, e.g., $Z^\alpha_{m_0,x} = \mathrm{artanh}(\frac{m_0-x}{m_0+x-2})$ for $x \in \mathbb{R}$ with $x \geq m_0$.

Proof of (a): using the addition theorem for the hyperbolic tangent function we prove that the function $h_{m_0} : [m_0, \infty) \to \mathbb{R}$, $x \mapsto R_{m_0,x}$ increases strictly on the interval $[\max(m_0, 4), \infty)$. Since $\lim_{x\to\infty} h_{m_0}(x) = \mathrm{artanh}(\frac{5}{8m_0-3})$ we obtain $R_{m_0,m_1} \leq \mathrm{artanh}(\frac{5}{8m_0-3})$ for all $m_1 \geq \max(m_0, 4)$.

Proof of (b): for $f_{m_0} : [m_0, \infty) \to \mathbb{R}$, $x \mapsto \sqrt{m_0}\, L_{m_0,x}$ we have $f'_{m_0}(x) > 0$ for $m_0 \leq x < 3m_0 - 4$, $f'_{m_0}(3m_0 - 4) = 0$ and $f'_{m_0}(x) < 0$ for $x > 3m_0 - 4$. Hence

$$f_{m_0}(x) \leq f_{m_0}(3m_0 - 4) \Leftrightarrow \sqrt{m_0}\, L_{m_0,x} \leq \sqrt{m_0}\, \mathrm{artanh}(\tfrac{1}{3m_0-4}) \text{ for } x \geq m_0.$$

The right hand side of this estimate is decreasing in m_0 and smaller than $\frac{\pi}{2}$ for $m_0 = 2$ and therefore smaller than $\frac{\pi}{2}$ for all $m_0 \geq 2$.

Proof of (c): $f'_{m_0}(x) < 0$ for $x \geq 3m_0 - 4$ and $\lim_{x\to\infty} f_{m_0}(x) = \mathrm{artanh}(\frac{1+\sqrt{17}}{16m_0-(17+\sqrt{17})})$ yield the claim. $\qquad\square$

Next we prove the following extended version of Theorem E.

Theorem 7.2: *Let r be a solution of the (m_0, m_1)-BVP with $m_0 \geq 2$. Then there exists $\ell_0 \in \{-1, 0, 1\}$ such that $(2\ell_0-1)\frac{\pi}{2} \leq r(x) \leq (2\ell_0+1)\frac{\pi}{2}$ for all $x \geq d^+_{m_0,m_1}$. Furthermore, $\lim_{x\to\infty} r(x) = \pm\frac{\pi}{2}$ or $\lim_{x\to\infty} r(x) = \pm\frac{3\pi}{2}$ and the Brouwer degree of the self-map ψ_r of $\mathbb{S}^{m_0+m_1+1}$ is ± 1 or ± 3.*

The strategy of the proof is as follows (considering the picture at the beginning of Section 3 helps to understand the idea):

- By Theorem 3.11 there exists a constant $d^-_{m_0,m_1} \in \mathbb{R}$ such that either $0 \leq r(x) \leq \pi$ or $-\pi \leq r(x) \leq 0$ for all $x \leq d^-_{m_0,m_1}$.
- By Lemma 3.7 there exists an integer $\ell_0 \in \mathbb{Z}$ such that $(2\ell_0 - 1)\frac{\pi}{2} \leq r(x) \leq (2\ell_0 + 1)\frac{\pi}{2}$ for all $x \geq d^+_{m_0,m_1}$.
- Since the first derivative of r is bounded we find $|r(d^-_{m_0,m_1}) - r(d^+_{m_0,m_1})| \leq \frac{\pi}{2}$, which implies $\ell_0 \in \{-1, 0, 1\}$. By Subsection 1 the Brouwer degree of r can thus only attain the values ± 1 or ± 3.

Proof. By Theorem 3.11 either $0 \leq r(x) \leq \pi$ or $-\pi \leq r(x) \leq 0$ for all $x \leq d^-_{m_0,m_1}$. Furthermore, Lemma 3.4 and Lemma 3.9 yield

$$|r(d^+_{m_0,m_1})| \leq \pi + \sqrt{m_0}\, L_{m_0,m_1} + \sqrt{m_1 + 1}\, (Z^\beta_{m_0,m_1} - Z^\alpha_{m_0,m_1}) + \sqrt{m_1}\, (d^+_{m_0,m_1} - Z^\beta_{m_0,m_1}).$$

For each $2 \leq m_0 \leq 5$ let $m_1^{max} \in \mathbb{N}$ be such that $|r(d^+_{m_0,m_1})| \leq \frac{3\pi}{2}$ for all m_1 with $m_0 \leq m_1 \leq m_1^{max}$. The following table gives m_1^{max} for $2 \leq m_0 \leq 5$. Next we sharpen the above estimate for $|r(d^+_{m_0,m_1})|$ by improving it on $I = [Z^\alpha_{m_0,m_1}, d^+_{m_0,m_1}]$. We may assume $r(Z^\alpha_{m_0,m_1}) \geq 0$ and $r'(x) \geq 0$ for $x \in I$, or similarly for $-r$, since otherwise the estimates become even better. Below we assume the first possibility.

m_0	2	3	4	5
m_1^{max}	4	27	60	106

TABLE 1. m_1^{max} for the cases $2 \leq m_0 \leq 5$

The (m_0, m_1)-ODE yields $r''(x) \leq \alpha_{m_0,m_1}(d^+_{m_0,m_1})r'(x) - \beta_{m_0,m_1}(x)\sin 2r(x)$ for all $x \in I$. By integrating once we thus obtain

$$(7.1) \quad r'(x) \leq r'(Z^\alpha_{m_0,m_1}) + \alpha_{m_0,m_1}(d^+_{m_0,m_1})\big(r(x) - r(Z^\alpha_{m_0,m_1})\big)$$
$$- \int_{Z^\alpha_{m_0,m_1}}^x \beta_{m_0,m_1}(\xi)\sin 2r(\xi)d\xi.$$

Lemma 3.9 and Lemma 7.1 imply $r(Z^\alpha_{m_0,m_1}) \leq \pi + \sqrt{m_0}\, L_{m_0,m_1} < \frac{3\pi}{2}$. In what follows we assume that there exists an $x_0 \in I$ such that $r(x_0) = \frac{3\pi}{2}$ and $r(x) < \frac{3\pi}{2}$ for all $x \in I_0 = \big[Z^\alpha_{m_0,m_1}, x_0\big]$. Consequently,

$$r'(x) \leq r'(Z^\alpha_{m_0,m_1}) + \alpha_{m_0,m_1}(d^+_{m_0,m_1})\big(\tfrac{3\pi}{2} - r(Z^\alpha_{m_0,m_1})\big) - \int_{Z^\alpha_{m_0,m_1}}^{Z^\beta_{m_0,m_1}} \beta_{m_0,m_1}(\xi)d\xi =: A,$$

for all $x \in I_0$. Thus $r(x) \leq r(Z^\alpha_{m_0,m_1}) + A(x - Z^\alpha_{m_0,m_1})$ for $x \in I_0$. Therefore (7.1) yields

$$r'(x) \leq r'(Z^\alpha_{m_0,m_1}) + \alpha_{m_0,m_1}(d^+_{m_0,m_1})A(x - Z^\alpha_{m_0,m_1}) - \int_{Z^\alpha_{m_0,m_1}}^{Z^\beta_{m_0,m_1}} \beta_{m_0,m_1}(\xi)d\xi$$

for $x \in I_0$. By integrating we thus obtain the following inequality for all $x \in I_0$

$$(7.2) \quad r(x) \leq r(Z^\alpha_{m_0,m_1}) + r'(Z^\alpha_{m_0,m_1})(x - Z^\alpha_{m_0,m_1}) + \tfrac{1}{2}\alpha_{m_0,m_1}(d^+_{m_0,m_1})A(x - Z^\alpha_{m_0,m_1})^2$$
$$- (x - Z^\alpha_{m_0,m_1})\int_{Z^\alpha_{m_0,m_1}}^{Z^\beta_{m_0,m_1}} \beta_{m_0,m_1}(\xi)d\xi.$$

In what follows we show that the right hand side of (7.2) is smaller than $\frac{3\pi}{2}$ for all $x \in I$, which contradicts the existence of x_0: the both inequalities $-\frac{1}{2} \leq \beta_{m_0,m_1}(Z^\alpha_{m_0,m_1})$, $\beta_{m_0,m_1}(d^+_{m_0,m_1}) \leq \frac{1}{2}$ and the fact that β increases strictly imply $|\beta_{m_0,m_1}(x)| \leq \frac{1}{2}$ for $x \in I$. Thus

$$- \int_{Z^\alpha_{m_0,m_1}}^{Z^\beta_{m_0,m_1}} \beta_{m_0,m_1}(\xi)d\xi \leq \tfrac{1}{2}\, R_{m_0,m_1}.$$

Since the right hand side of (7.2) is strictly increasing in x it is sufficient to prove $r(d^+_{m_0,m_1}) < \frac{3\pi}{2}$. Using $\alpha_{m_0,m_1}(d^+_{m_0,m_1}) \leq \frac{5}{4}$ and $A \geq 0$, inequality (7.2) implies

$$r(d^+_{m_0,m_1}) \leq r(Z^\alpha_{m_0,m_1}) + \sqrt{m_0}\, R_{m_0,m_1} + \tfrac{1}{2}\, R^2_{m_0,m_1}(1 + \tfrac{5}{4}A).$$

By Lemma 3.9 and the above considerations we have

$$(7.3) \qquad A \leq \sqrt{m_0} + \tfrac{5}{4}\big(\tfrac{3\pi}{2} - r(Z^\alpha_{m_0,m_1})\big) + \tfrac{1}{2}\, R_{m_0,m_1}.$$

Combining the two preceding estimates we get

$$(7.4) \quad r(d^+_{m_0,m_1}) \leq \sqrt{m_0}\, R_{m_0,m_1} + r(Z^\alpha_{m_0,m_1})(1 - \tfrac{5^2}{2^5} R^2_{m_0,m_1})$$
$$+ \tfrac{1}{2} R^2_{m_0,m_1}(1 + \tfrac{5}{4}\sqrt{m_0} + \tfrac{3\pi 5^2}{2^5} + \tfrac{5}{2^3} R_{m_0,m_1}).$$

By Lemma 7.1 the coefficient of $r(Z^\alpha_{m_0,m_1})$ is non-negative for $m_1 \geq \max(m_0, 4)$. Furthermore, Lemma 7.1 implies $r(Z^\alpha_{m_0,m_1}) \leq \pi + \sqrt{m_0}\, \mathrm{artanh}(\frac{1}{3m_0-4}) < \frac{3\pi}{2}$. Therefore

$$r(d^+_{m_0,m_1}) \leq \sqrt{m_0}\, R_{m_0,m_1} + \tfrac{1}{2} R^2_{m_0,m_1}\big(1 + \tfrac{5}{4}\sqrt{m_0} + \tfrac{\pi 5^2}{2^5}$$
$$- \tfrac{5^2}{2^4}\sqrt{m_0}\, \mathrm{artanh}(\tfrac{1}{3m_0-4}) + \tfrac{5}{2^3} R_{m_0,m_1}\big) + \pi + \sqrt{m_0}\, \mathrm{artanh}(\tfrac{1}{3m_0-4}).$$

Case 1: $m_0 \geq 3$. Since in the preceding inequality the expression in the bracket after $R^2_{m_0,m_1}$ is non-negative, Lemma 7.1 yields

$$r(d^+_{m_0,m_1}) \leq \pi + \sqrt{m_0}\, \mathrm{artanh}(\tfrac{23}{24m_0-17}) + \tfrac{1}{2}\mathrm{artanh}(\tfrac{5}{8m_0-3})^2\big(1 + \tfrac{5}{4}\sqrt{m_0} + \tfrac{\pi 5^2}{2^5}$$
$$- \tfrac{5^2}{2^4}\sqrt{m_0}\, \mathrm{artanh}(\tfrac{1}{3m_0-4}) + \tfrac{5}{2^3}\mathrm{artanh}(\tfrac{5}{8m_0-3})\big)$$

for $m_1 \geq \max(m_0, 4)$ and $m_0 \geq 3$, where we also use the addition theorem for artanh. From $r(d^+_{3,m_1}) < \frac{3\pi}{2}$ and the fact that the right hand side of the preceding inequality is decreasing in m_0 we get $r(d^+_{m_0,m_1}) < \frac{3\pi}{2}$ for $m_0 \geq 3$ and $m_1 \geq 4$, which contradicts our assumption. Hence there does not exist a point $x_0 \in I$ with $r(x_0) = \frac{3\pi}{2}$. Similarly, we prove that there cannot exist a point $x_1 \in I$ with $r(x_1) = -\frac{3\pi}{2}$. We thus obtain: for $m_0 \geq 3$ there exists $\ell_0 \in \{-1, 0, 1\}$ such that $(2\ell_0 - 1)\frac{\pi}{2} \leq r(x) \leq (2\ell_0 + 1)\frac{\pi}{2}$ for all $x \geq d^+_{m_0,m_1}$. Note that the case $m_0 = m_1 = 3$ is covered by Table 1.

Case 2: $m_0 = 2$. In what follows we restrict ourselves to $m_1 \geq 4$ since Table 1 covers the cases $m_1 = 2$ and $m_1 = 3$.

First we assume $r(Z^\alpha_{2,m_1}) \leq \pi$. By (7.3) we have $A \leq \sqrt{2} + \tfrac{5}{4}(\tfrac{3\pi}{2} - r(Z^\alpha_{2,m_1})) + \tfrac{1}{2} R_{2,m_1}$. Thus $r(d^+_{2,m_1}) \leq r(Z^\alpha_{2,m_1}) + A R_{2,m_1}$ yields

$$r(d^+_{2,m_1}) \leq r(Z^\alpha_{2,m_1}) + \big(\sqrt{2} + \tfrac{5}{4}(\tfrac{3\pi}{2} - r(Z^\alpha_{2,m_1})) + \tfrac{1}{2} R_{2,m_1}\big) R_{2,m_1}.$$

One proves easily that the resulting coefficient of $r(Z^\alpha_{2,m_1})$, namely $1 - \tfrac{5}{4} R_{2,m_1}$, is non-negative for all $m_1 \geq 2$. Thus we may assume $r(Z^\alpha_{2,m_1}) = \pi$. Consequently,

$$r(d^+_{2,m_1}) \leq \pi + \big(\sqrt{2} + \tfrac{5\pi}{2^3} + \tfrac{1}{2} R_{2,m_1}\big) R_{2,m_1}.$$

Using part (i) of Lemma 7.1 we get $r(d^+_{2,m_1}) < \frac{3\pi}{2}$ for all $m_1 \geq 4$. However, this contradicts our assumption that there exist a point $x_0 \in I$ with $r(x_0) = \frac{3\pi}{2}$.

Next we assume $r(Z^\alpha_{2,m_1}) \geq \pi$. Since $\lim_{x \to -\infty} V^r_{2,m_1}(x) = 1$ and V^r_{2,m_1} decreases on the interval $(-\infty, Z^\alpha_{2,m_1}]$ we have

$$(7.5) \qquad r'(x)^2 \leq 2 + 2\beta_{2,m_1}(x)\cos^2(r(x)),$$

for all $x \in (-\infty, Z^\alpha_{2,m_1}]$. From this we obtain an upper bound for $r'(Z^\alpha_{2,m_1})$.

We may assume $r(Z^\alpha_{2,m_1}) = \pi + \sqrt{2} L_{2,m_1}$: suppose that $r(Z^\alpha_{2,m_1})$ attains a smaller value $\widetilde{r}(Z^\alpha_{2,m_1})$ between π and $\frac{3\pi}{2}$, i.e., $r(Z^\alpha_{2,m_1}) = \widetilde{r}(Z^\alpha_{2,m_1}) + \Delta r$ with $\pi \leq \widetilde{r}(Z^\alpha_{2,m_1}) < \frac{3\pi}{2}$ and $\Delta r > 0$. Since $\beta_{2,m_1}(Z^\alpha_{2,m_1}) \leq 0$ the upper bound for $r'(Z^\alpha_{2,m_1})$ becomes smaller, while A increases by $\alpha_{2,m_1}(d^+_{2,m_1})\Delta r$. If we neglect the fact that the upper bound

for $r'(Z_{2,m_1}^\alpha)$ becomes smaller, the right hand side of inequality (7.2) changes by $c :=$ $(\frac{1}{4}\alpha_{2,m_1}^2(d_{2,m_1}^+)R_{2,m_1}^2 - 1)\Delta r$. Since $\alpha_{2,m_1}(d_{2,m_1}^+) \leq \frac{5}{4}$ for all $m_1 \geq 2$, the first statement of Lemma 7.1 implies $c < 0$ for $m_1 \geq 4$. In other words, in these cases the estimate (7.2) becomes even better.

Plugging $r(Z_{2,m_1}^\alpha) = \pi + \sqrt{2}L_{2,m_1}$ into (7.5) and using Lemma 7.1 (ii) we obtain the inequality $|r'(Z_{2,m_1}^\alpha)| \leq u_{m_1}$, where $u_{m_1} := \left(2 + \frac{2-m_1}{m_1}\cos^2(\operatorname{artanh}(\frac{1}{2})\sqrt{2})\right)^{\frac{1}{2}}$.

We now proceed similar as above, where we use the estimate $|r'(Z_{2,m_1}^\alpha)| \leq u_{m_1}$ instead of $|r'(Z_{2,m_1}^\alpha)| \leq \sqrt{2}$. Consequently, instead of (7.4) we obtain

$$r(d_{2,m_1}^+) \leq u_{m_1}R_{2,m_1} + r(Z_{2,m_1}^\alpha)(1 - \frac{5^2}{2^5}R_{2,m_1}^2) + \frac{1}{2}R_{2,m_1}^2(1 + \frac{5}{4}u_{m_1} + \frac{3\pi 5^2}{2^5} + \frac{5}{2^3}R_{2,m_1}).$$

Next we find an upper estimate for $r(Z_{2,m_1}^\alpha)$: we may assume $r' \geq 0$ on $I_1 = [d_{2,m_1}^-, Z_{2,m_1}^\alpha]$ since otherwise the estimates become even better. From $r' \leq \sqrt{2}$ on I_1 and Theorem 3.11 we deduce $r(x) \leq \pi + \sqrt{2}(x - d_{2,m_1}^-)$ for all $x \in I_1$. Hence (7.5) implies

$$r'(x) \leq (2 + 2\beta_{2,m_1}\cos^2(\sqrt{2}(x - d_{2,m_1}^-)))^{\frac{1}{2}} =: v_{m_1}(x)$$

for all $x \in I_1$. This result together with Theorem 3.11 implies

$$r(Z_{2,m_1}^\alpha) \leq \pi + \int_{d_{2,m_1}^-}^{Z_{2,m_1}^\alpha} v_{m_1}(x)\,dx =: w_{m_1}.$$

Substituting this into the preceding estimate for $r(d_{2,m_1}^+)$ yields

$$r(d_{2,m_1}^+) \leq u_{m_1}R_{2,m_1} + w_{m_1} + \frac{1}{2}R_{2,m_1}^2(1 + \frac{5}{4}u_{m_1} + \frac{3\pi 5^2}{2^5} + \frac{5}{2^3}R_{2,m_1} - \frac{5^2}{2^4}w_{m_1}).$$

Since the expression in the bracket after R_{2,m_1}^2 is positive for all $m_1 \geq 2$ we can apply Lemma 7.1 and thus obtain

$$r(d_{2,m_1}^+) \leq u_{m_1}\operatorname{artanh}(\tfrac{5}{13}) + w_{m_1}$$
$$+ \tfrac{1}{2}\operatorname{artanh}(\tfrac{5}{13})^2(1 + \tfrac{5}{4}u_{m_1} + \tfrac{3\pi 5^2}{2^5} + \tfrac{5}{2^3}\operatorname{artanh}(\tfrac{5}{13}) - \tfrac{5^2}{2^4}w_{m_1}).$$

For $m_1 = 3$ the right hand side of this inequality is smaller than $\frac{3\pi}{2}$. Furthermore, it is decreasing in m_1. Consequently, we get $r(d_{2,m_1}^+) < \frac{3\pi}{2}$ for $m_1 \geq 3$. However, this contradicts our assumption that there exist a point $x_0 \in I$ with $r(x_0) = \frac{3\pi}{2}$.

Since the case $m_0 = m_1 = 2$ is covered by Table 1 we thus obtain: for $m_0 = 2$ and each $m_1 \geq 2$ there exists an integer $\ell_0 \in \{-1, 0, 1\}$ such that $(2\ell_0 - 1)\frac{\pi}{2} \leq r(x) \leq (2\ell_0 + 1)\frac{\pi}{2}$ for all $x \geq d_{m_0,m_1}^+$. $\qquad\square$

Remark 7.3: The result of Theorem 7.2 is optimal in the sense that numerical results show that for each $\ell_0 \in \{-1, 0, 1\}$ there exist solutions of the (m_0, m_1)-ODE with $\lim_{x\to-\infty} r(x) = 0$ and $(2\ell_0 - 1)\frac{\pi}{2} \leq r(x) \leq (2\ell_0 + 1)\frac{\pi}{2}$ for $x \geq d_{m_0,m_1}^+$. However, we only found solutions with $\lim_{x\to\infty} r(x) = \pm\frac{\pi}{2}$, i.e., solutions with Brouwer degree ± 1.

Question 2: Do all solutions of the (m_0, m_1)-BVP have Brouwer degree ± 1?

7.2. **Brouwer degree for large initial velocities.** In what follows we show that the Brouwer degree of r_v is given by ± 1 if we chose v "sufficiently large" and thereby establish Theorem F.

Theorem 7.4: *For $m_1 \geq 2$ there exists $v_0 > 0$ such that each solution r_v of the (m_0, m_1)-BVP with $v \geq v_0$ has Brouwer degree ± 1.*

Proof. Let T be as in the proof of Theorem 6.3, $\epsilon > 0$ and $\mu \in [\frac{1}{2}, 1)$. Lemma 6.2 implies the existence of $v_0 \in \mathbb{R}$ such that $v \geq v_0$ yields $\rho_v(x) < \epsilon$ for $d^-_{m_0, m_1} \leq x \leq T$. Thus

$$(7.6) \qquad |\varphi_v(x)| < \epsilon \text{ for } v \geq v_0 \text{ and } d^-_{m_0, m_1} \leq x \leq T.$$

Furthermore, the proof of Theorem 6.3 yields $W^{r_v}_{m_0, m_1}(T) \geq \mu \frac{m_1}{2}$ for all $v \geq v_0$ and

$$(7.7) \qquad W^{r_v}_{m_0, m_1}(x) - W^{r_v}_{m_0, m_1}(T) \leq (1 - \mu) \frac{m_1}{2} \text{ for all } x \geq T, v \geq v_0.$$

In what follows we may assume $\epsilon < \pi$. Hence by combining inequality (7.6) and Lemma 3.7 we have that each solution r_v of the (m_0, m_1)-BVP with $v \geq v_0$ satisfies either (i) $\lim_{x \to \infty} r_v(x) = \frac{3\pi}{2}$ or (ii) $\lim_{x \to \infty} r_v(x) = -\frac{\pi}{2}$ or (iii) $\lim_{x \to \infty} r_v(x) = \frac{\pi}{2}$.

Below we prove that the first two cases cannot occur if the initial velocity is chosen big enough: let $v \geq v_0$ be given. By (7.6) the choice $\epsilon = \frac{\pi}{2}$ yields $0 < r_v(T) < \pi$. Assume either (i) or (ii). In each of theses cases there exist $x_2 \geq T$ and $k_0 \in \{0, 1\}$ such that $r_v(x_2) = k_0 \pi$. By Lemma 3.3 we get $\frac{1}{2} r_v'(x_2)^2 = W^{r_v}_{m_0, m_1}(x_2) \geq W^{r_v}_{m_0, m_1}(T) \geq \mu \frac{m_1}{2}$. Hence

$$\frac{d}{dx} W^{r_v}_{m_0, m_1}(x_2) = \alpha_{m_0, m_1}(x_2) r_v'(x_2)^2 \geq \mu \frac{m_1(m_1 - 1)}{2}.$$

Lemma 3.4 and Lemma 3.9 imply that the absolute value of the second derivative of $W^{r_v}_{m_0, m_1}$ is bounded. Consequently, there exists an point $x_3 > x_2$ which depends on μ, m_0 and m_1 only such that

$$\frac{d}{dx} W^{r_v}_{m_0, m_1}(x) \geq \mu \frac{m_1(m_1 - 1)}{4}$$

for all $x \in [x_2, x_3]$. Thus integrating yields

$$W^{r_v}_{m_0, m_1}(x_3) - W^{r_v}_{m_0, m_1}(x_2) \geq \mu \frac{m_1(m_1 - 1)}{4}(x_3 - x_2) > 0.$$

Therefore we have

$$W^{r_v}_{m_0, m_1}(x_3) - W^{r_v}_{m_0, m_1}(T) \geq W^{r_v}_{m_0, m_1}(x_2) - W^{r_v}_{m_0, m_1}(T) + \mu \frac{m_1(m_1 - 1)}{4}(x_3 - x_2)$$
$$\geq \mu \frac{m_1(m_1 - 1)}{4}(x_3 - x_2) > 0.$$

Clearly, there exists a constant $c \in \mathbb{R}$ such that $x_3 - x_2 > c > 0$ for all $\mu \in [\frac{1}{2}, 1)$. If we choose μ sufficiently near to 1 we thus obtain a contradiction to (7.7). $\qquad \square$

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